Theory of Interest Rates in Markets Modeled by Jump Diffusion Processes

by

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Abstract

The object of this thesis is to study the classical Heath-Jarrow-Morton(HJM) model for interest rates, and the corresponding London Interbank Offered Rate(LIBOR) model, when the noise is driven by an Itô-Lévy process instead of only a Brownian motion. When the model is driven by only a Brownian motion we have well known theory concerning the risk neutral measures and how to compute arbitrage free prices for options. We will find corresponding results when the market is modeled by jump diffusions. One of the problems with markets modeled by jump diffusions is that these models will in general be incomplete, so we will get several equivalent local martingale measures(ELMM), so one of the problems we will look at is how to find such measures. Next we will look at how to compute the price of a European call option for a general ELMM, this will be done with the use of Fourier transforms and computation of a characteristic function. At last we will look at a utility maximization problem, and how to find investment strategies for this, and one of the methods we will use to find this is a duality method.

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At last I would like to thank my friends and family for the support, positive feedback and encouragement they have given me during my studies, and especially the last couple of months.

Notation

In this thesis we will use several abbreviations for phrases that are commonly used, most of them should be well known, but we will list them here and write a comment where it is needed, we will also write it out in full the first time it is used in the actual thesis.

Heath-Jarrow-Morton: HJM, our original zero coupon bond model, we will use the same abbreviation in Chapter 2, 4 and 5, but for slightly different models, but what we refer to will be clear from context.

London Interbank Offered Rate: LIBOR, the forward interest rate model derived from our HJM model, we will also here use the same abbreviation for slightly different models in the different chapters.

Equivalent Local Martingale Measure: ELMM, will be used for measures which our discounted bond price is a local martingale under, we also use martingale measure at times, since that is what we end up with.

Minimal Entropy Martingale Measure: MEMM, a special kind of martingale measure we find in Chapter 5.

We will also use the term "equation for no arbitrage", and by this we mean the equation we get from the Girsanov theorem, that must hold if the new measure shall be an ELMM.

When it comes to mathematical notation we will use $\mathcal{M}$ for the set of ELMM's and $\mathcal{A}$ for the set of admissible trading strategies. The requirements for a trading strategy to be admissible is standard, it must be self-financing, $\mathcal{F}_t$-predictable and our wealth process must be lower bounded.

In the thesis we will also use the terms Minimal Entropy Martingale Measure, Esscher transform, Compound Return Esscher Transformed Martingale Measure and Simple Return Esscher Transformed Martingale Measure for Itô-Lévy processes. These terms are taken from the equivalent expressions for Lévy processes, I have done this since I have not found anything like this described for Itô-Lévy processes.

We will also have several integrals in this thesis, and for our integrands we will in general write $\sigma(t)$ instead of $\sigma(t, \omega)$, even if our integrands are stochastic. In some cases we will simplify our expressions and assume we will actually have deterministic integrands, but we will specify it when needed.

Contents

1 Introduction

1.1 Backgrond

The background for this thesis is the well-established HJM model and the corresponding LIBOR model driven by a Brownian motion. In this thesis we will try to extend the results from this model to a model where the noise is driven by jump diffusions. The reasons for why we are interested in this extended model are many. First of all, if we model it with only a Brownian motion we will get a complete marked, and while being quite nice for computations, since it means all claims can be replicated, it is not necessarily realistic from an economical point of view. Also a model driven by a Brownian motion will be continuous, and interest rates, as all other risky assets, will not have continuous prices, prices will in general jump when something unexpected happens in the financial markets, like the bursting of the U.S housing bubble[1] a few years back, and having a model that can simulate such movements will be more realistic then continuous ones. At last we will see that if we do not have deterministic integrands in our model, computations will be hard, and we will be quite bonded in what kind of probability distributions our model will have if we only work with a Brownian motion, and working with Lévy processes will give us a lot more freedom in that aspect.

1.2 Known Results

As said earlier, the background for this thesis is the HJM model and the corresponding LIBOR-model, and in Chapter 2 we go through known results from this. We start by introducing the HJM model, and then we find the risk neutral measure for this. After that we move on to the LIBOR model and we find what is called a T-forward measure, which is the risk neutral measure for our LIBOR model. At last we compute the price of a European call option when the underlying is the LIBOR. When we compute the option price we need the result that the Itô-integral $\int_0^t f(s)dB(s)$, where $f(s)$ is deterministic, is normally distributed, so we have included a proof for this.

In Chapter 3 we go through theoretical background for Itô-Lévy processes, and most of it, if not all, should be known. The first chapters go through standard theory about Lévy processes and Itô-Lévy processes to show the difference between these and the Brownian motion. Later we introduce the

[1] This will also simulate smaller jumps that happen all the time for economical markets

Esscher transform, and while the standard Esscher transform for Lévy processes are known, I have not seen any papers using my Esscher like transform for Itô-Lévy processes, where I introduce a Radon-Nikodym derivative as $\frac{\exp(\int_0^t \theta(s)dX(s))}{E[\exp(\int_0^t \theta(s)dX(s))]}$, this is not to say it is not done before. We also find a characteristic function for Itô-Lévy processes, and while it should be known theory, the proof is my own, but inspired by known proofs of the Lévy-Khintchine formula. In the last sections we go through known stuff about the maximum principle, and how option prices can be found by Fourier transforms and characteristic functions. At last we go through a duality method that shows the connection between optimal investment strategies and optimal martingale measures. For readers experienced in the topics of interest rate modeling and Lévy processes, these chapters can be skipped.

1.3 My Contributions

The new theory presented in this thesis is shown in Chapter 4 and 5. In Chapter 4 we introduce the HJM and corresponding LIBOR model driven by jump diffusions, and we find conditions that need to hold for our set of ELMM's. In Chapter 5 we first focus on finding different martingale measures, and our main result here is showing the equality between the Minimal Entropy Martingale Measure and the Simple Return Esscher Transformed Martingale measure, an equality which is known for Lévy processes, but as far as I know not for Itô-Lévy processes. In Section 5.4 we look at the price of a European call option and we show how this can be calculated using Fourier transforms. At last we look at investment strategies for a utility maximization problem, and we compute this by both using the duality method and with the maximum principle.

2 The HJM and LIBOR Model

In this chapter we will go through known stuff from interest rate modeling with Brownian motions. We will start with the classical Heath-Jarrow-Morton model, and introduce the risk neutral measure for this, and then we will introduce the forward LIBOR-model and the T-forward measure. At last we will show that the Itô-integral $\int_0^t f(s)dB(s)$, where $f(s)$ is deterministic, is normally distributed, and we will use this to compute an option price where our underlying is the forward LIBOR-rates.

The theory for this chapter is in general taken from [5]. Most of the theory is also presented in a classical paper by Brace, Gatarek and Musiela [9], but in a bit more general and technical setting.

2.1 The Heath-Jarrow-Morton model

In this section we will introduce the Heath-Jarrow-Morton model(from now on HJM), which is a interest rate model, driven by a Brownian motion. In the HJM framework, we model the interest rates $f(t,s)$ over the infinitesimal interval $[s, s+\Delta s]$, as seen from the timepoint $t \leq s$, and is therefore seen as an instantaneous forward curve, and the interest rates $f(t,s)$ are modeled by:

$$f(t,s) = f(0,s) + \int_0^t \alpha(v,s)dv + \int_0^t \sigma(v,s)dB(v) \tag{2.1}$$

Here $B(t)$ is a classical Brownian motion. For the integrals in (2.1) to make sense, we need $\alpha(v,s)$ and $\sigma(v,s)$ to be $\mathcal{F}_v$-measurable for all $s \geq t$, this will also make $f(t,s)$, $\mathcal{F}_t$-measurable. We will also assume that $E\left[\int_0^s \alpha^2(v,s)dv\right] < \infty$, and $E\left[\int_0^s \sigma^2(v,s)dv\right] < \infty$ for all $s \leq T$.

Now we have our interest rates, and we can then define the price of a zero coupon bond, with maturity T, by:

$$P(t,T) = \exp\left(-\int_t^T f(t,s)ds\right) \tag{2.2}$$

From our previous assumptions we have that $f(t,s)$ is $\mathcal{F}_t$-measurable for $s \geq t$ so it will also be $\mathcal{F}_s$-measurable, and our expression (2.2) will be well defined.

Later on we will define options for this bond, more specifically a European call option, and the general idea behind option pricing, is to find an arbitrage free price for the option, which is done by finding a probability measure $Q(\omega)$, under which the discounted bond price, denoted $\tilde{P}(t,T)$, and given by:

$$\tilde{P}(t,T) = \frac{P(t,T)}{\beta(t)} \tag{2.3}$$

is a local martingale. Here $\beta(t)$ is the compounded interest rates over the interval $[0,t]$:

$$\beta(t) = \exp\left(\int_0^t r(s)ds\right) \tag{2.4}$$

where $r(s) = f(s,s)$, is the observed interest rates at time s.

2.2 Equivalent Local Martingale Measure for the HJM Model

The next step will then be to find an equivalent local martingale measure(ELMM) for $\tilde{P}(t,T)$. This will be done by finding the dynamics $d\tilde{P}(t,T)$, then we can define a probability measure Q, and a corresponding Brownian motion $\tilde{B}(t) = B(t) + \int_0^t q(t)dt$, where $q(t)$ will nullify the drift term from $d\tilde{P}(t,T)$. The first thing we will do is write out the expression for $P(t,T)$:

$$\begin{aligned} P(t,T) &= \exp\left(-\int_t^T f(t,s)ds\right) \\ &= \exp\left(-\int_t^T f(0,s)ds + \int_t^T \int_0^t \alpha(v,s)dvds\right. \\ &\quad \left. + \int_t^T \int_0^t \sigma(v,s)dB(v)ds\right) \end{aligned} \tag{2.5}$$

Since what we want is to find $d\tilde{P}(t,T)$, where our differential is taken with respect to t, we need to use Itô's-formula, but since t is a factor in both our integrals this will not be straight forward. Our solution to this will be to add and subtract some terms, and then use a stochastic Fubini's theorem to get integrals where t is only a factor in the outer integral.

Since $-f(t,s) = f(s,s) - f(t,s) - f(s,s)$, we can rewrite (2.5) like this:

$$
\begin{aligned}
P(t,T) &= \exp\left(\int_t^T f(s,s) - f(t,s)ds - \int_t^T f(s,s)ds\right) \\
&= \exp\left(-\int_t^T f(s,s)ds + \int_t^T \left(\int_0^s \alpha(v,s)dv - \int_0^t \alpha(v,s)dv\right) ds\right. \\
&\quad \left. + \int_t^T \left[\int_0^s \sigma(v,s)dB(v) - \int_0^t \sigma(v,s)dB(v)\right] ds\right) \\
&= \exp\left(-\int_t^T f(s,s)ds + \int_t^T \int_t^s \alpha(v,s)dvds\right. \\
&\quad \left. + \int_t^T \int_t^s \sigma(v,s)dB(v)ds\right)
\end{aligned}
\tag{2.6}
$$

Now we shall use a stochastic Fubini's theorem to interchange the limits in these integrals.

Theorem 1. Stochastic Fubini's theorem
The stochastic Fubini theorem says that if we have a probability space $(\Omega, \mathcal{F}_T, \mathbb{P})$, a process $\phi : [0,S] \times [0,T] \times \Omega \to \mathbb{R}$, that is well defined. Then if we integrate this with respect to a semimartingale X and a measure μ we can change the order of integration with respect to X and μ if this inequality holds:

$$\int_0^S \left| \int_0^T \phi(s,t)dX(t) \right| d\mu(s) < \infty \tag{2.7}$$

and we get this:

$$\int_0^S \int_0^T \phi(s,t)dX(t)d\mu(s) = \int_0^T \int_0^S \phi(s,t)d\mu(ds);\ a.s \tag{2.8}$$

Proof. See [8] for all requirements on the processes and the integrals. □

This theorem tells us that we can change the order of integration in our stochastic integrals under some conditions. The integrability condition will hold since we have assumed square integrability on our processes. To see this we use Hölder's inequality like this

$$\int_0^T |f(s) \cdot 1| ds \leq \int_0^T |f(s)|^2 ds \int_0^T 1^2 ds = \int_0^T |f(s)|^2 ds \cdot T < \infty \tag{2.9}$$

If we use the stochastic Fubini's theorem on (2.6) to interchange the limits, we will get this:

$$P(t,T) = \exp\left(-\int_t^T f(s,s)ds + \int_t^T \int_v^T \alpha(v,s)dsdv + \int_t^T \int_v^T \sigma(v,s)dsdB(v)\right)$$

Now we can use this to find our discounted bond price, defined by (2.3), and it will be given by:

$$\tilde{P}(t,T) = \exp\left(-\int_0^T f(s,s)ds + \int_t^T \int_v^T \alpha(v,s)dsdv + \int_t^T \int_v^T \sigma(v,s)dsdB(v)\right)$$

Since t is now only a factor in the outer integral, we can use Itô's formula directly, to obtain an expression for $d\tilde{P}(t,T)$, like this:

$$\begin{aligned} d\tilde{P}(t,T) &= \tilde{P}(t,T)\left[\left(\frac{1}{2}\left(\int_t^T \sigma(t,s)ds\right)^2 \right.\right. \\ &\quad \left.\left. - \int_t^T \alpha(t,s)ds\right) dt - \int_t^T \sigma(t,s)dsdB(t)\right] \end{aligned} \tag{2.10}$$

The next thing we would like to do, is to find a new probability measure Q, and a Brownian motion for this probability measure, such that $\tilde{P}(t,T)$ is a local martingale under this new measure. To do this we will have to find a process $\tilde{B}(t) = \int_0^t q(s)ds + B(t)$, which will be a Brownian motion under our new probability measure, and we want $q(t)$ to nullify the drift term, $(\cdots)dt$, from our expression of $d\tilde{P}(t,T)$. If we substitute for our new Brownian motion into (2.10) we will get:

$$\begin{aligned} d\tilde{P}(t,T) &= \tilde{P}(t,T)\left[\left(\frac{1}{2}\left(\int_t^T \sigma(t,s)ds\right)^2 \right.\right. \\ &\quad \left. - \int_t^T \alpha(t,s)ds\right) dt - \int_t^T \sigma(t,s)dsd\tilde{B}(t) \\ &\quad \left. + \int_t^T \sigma(t,s)q(t)dsdt\right] \end{aligned} \tag{2.11}$$

which means that to remove the drift, $q(t)$ has to satisfy this equation:

$$\int_t^T \alpha(t,s)ds = \frac{1}{2}\left(\int_t^T \sigma(t,s)ds\right)^2 + q(t)\int_t^T \sigma(t,s)ds \tag{2.12}$$

for $\tilde{P}(t,T)$ to be a local martingale under the new probability measure Q, which $\tilde{B}(t)$ is a Brownian motion under. We also see that $q(t)$ will be given by functions that are defined in the interval $[t,T]$, but we know from Girsanov's theorem that $q(t)$ need to be $\mathcal{F}_t$-adapted, but since we have already assumed that $\alpha(t,s)$ and $\sigma(t,s)$ are $\mathcal{F}_t$-measurable for $t \leq s$, we don't need any extra conditions for this to hold. To simplify the condition for no arbitrage, we differentiate with respect to T on both sides of (2.12), and we get this expression for $\alpha(t,T)$:

$$\alpha(t,T) = \sigma(t,T) \cdot (\bar{\sigma}(t,T) + q(t)) \tag{2.13}$$

Here $\bar{\sigma}(t,T) = \int_t^T \sigma(t,s)ds$, and $\bar{\sigma}(t,T)$ will also be $\mathcal{F}_t$-measurable . Our risk-premium, $q(t)$, is then given by $q(t) = \frac{\alpha(t,T)}{\sigma(t,T)} - \bar{\sigma}(t,T)$, and we can define our new probability measure Q, by $q(t)$. From Girsanov's theorem[2] we get $Q(A) = E[\mathbb{1}(A) \cdot Z(T)]$, where $Z(T)$ is given by:

$$Z(t) = \exp\left(-\int_0^t q(s)dB(s) - \frac{1}{2}\int_0^t q(s)^2 ds\right) \tag{2.14}$$

Now we have an uniquely given martingale measure, which means that our bond price market is complete when the noise of the forward rates comes from one Brownian motion, this makes it easy to price and hedge options based on this bond. We also get under this measure that our forward rates are given by:

$$f(t,s) = f(0,s) + \int_0^t \sigma(v,s)\bar{\sigma}(v,s)dv + \int_0^t \sigma(v,s)d\tilde{B}(v) \tag{2.15}$$

As we see here, our forward rates are only dependent of $\sigma(\cdot,\cdot)$ under our risk neutral measure Q. This means we can specify a $\mathcal{F}_t$-adapted volatility process $\sigma(t,\cdot)$, then specify a risk-premium $q(t)$, which is $\mathcal{F}_t$-adapted, and from this a process $\alpha(t,\cdot)$ is given by (2.13)

2.3 The LIBOR Model

In the LIBOR model(London interbank offered rate), we model our forward rates by a log-normal distributed process instead of the forward rated rates

[2]We assume Girsanov's theorem for Brownian motions are known, if not, look in Chapter 3 where a Girsanov's theorem for the more general Lévy process is defined

given in our previous section. The reason for why we want a log-normal distributed process is that then we can compute option prices in the same way as in the Black-Scholes model.

The idea of the LIBOR-model is to have a short position of size 1 in T maturity bonds, $P(t,T)$, and a long position in $T+\delta$ maturity bonds, $P(t,T+\delta)$, of size $\frac{P(t,T)}{P(t,T+\delta)}$, this corresponds to LIBOR rates, $L(t,T)$, given by:

$$1+\delta L(t,T)=\frac{P(t,T)}{P(t,T+\delta)}$$

which is equivalent to:

$$L(t,T)=\frac{P(t,T)-P(t,T+\delta)}{\delta P(t,T+\delta)}$$

Here δ is the tenor[3] of the LIBOR, and is usually a small number, like 0.25 years. Note that $P(t,T)>P(t,T+\delta)>0$, so $L(t,T)>0$.

The problem now is first to find the dynamics of $L(t,T)$, then we will find a new probability measure, such that $L(t,T)$ will be a martingale under this measure. The first thing we will do, is to compute $\frac{P(t,T)}{P(t,T+\delta)}$:

$$\begin{aligned}
\frac{P(t,T)}{P(t,T+\delta)} &= \exp\left(\int_T^{T+\delta} f(t,s)ds\right) \\
&= \exp\left(\int_T^{T+\delta} f(0,s)ds+\int_T^{T+\delta}\int_0^t \sigma(v,s)\bar{\sigma}(v,s)dvds\right. \\
&\quad \left.+\int_T^{T+\delta}\int_0^t \sigma(v,s)d\tilde{B}(v)ds\right) \\
&\overset{\text{Fubini}}{=} \exp\left(\int_T^{T+\delta} f(0,s)ds+\int_0^t\int_T^{T+\delta} \sigma(v,s)\bar{\sigma}(v,s)dsdv\right. \\
&\quad \left.+\int_0^t\int_T^{T+\delta} \sigma(v,s)dsd\tilde{B}(v)\right) \qquad (2.16)
\end{aligned}$$

and now we can easily compute $dL(t,T)$:

[3]Tenor is the time to maturity of a bond

$$
\begin{aligned}
dL(t,T) &= d\left(\frac{1}{\delta}\frac{P(t,T)}{P(t,T+\delta)} - \frac{1}{\delta}\right) \\
&= \frac{1}{\delta}\frac{P(t,T)}{P(t,T+\delta)}\left[\int_T^{T+\delta}\sigma(t,s)\bar{\sigma}(t,s)dsdt + \frac{1}{2}\left(\int_T^{T+\delta}\sigma(t,s)ds\right)^2 dt\right. \\
&\quad \left. + \int_T^{T+\delta}\sigma(t,s)dsd\tilde{B}(t)\right] \qquad (2.17)
\end{aligned}
$$

If we then note that $\int_T^{T+\delta}\sigma(t,s)ds = \int_t^{T+\delta}\sigma(t,s)ds - \int_t^T\sigma(t,s)ds = \bar{\sigma}(t,T+\delta)-\bar{\sigma}(t,T)$, and that $\int_T^{T+\delta}\sigma(t,s)\bar{\sigma}(t,s)ds = \frac{1}{2}(\bar{\sigma}^2(t,T+\delta)-\bar{\sigma}^2(t,T))$, we get this formula for $dL(t,T)$:

$$
\begin{aligned}
dL(t,T) &= \left(\frac{1}{\delta} + L(t,T)\right)\left[\bar{\sigma}(t,T+\delta)(\bar{\sigma}(t,T+\delta) - \bar{\sigma}(t,T))\right. \\
&\quad \left. +(\bar{\sigma}(t,T+\delta) - \bar{\sigma}(t,T))d\tilde{B}(t)\right] \\
&= \gamma(t,T)L(t,T)\bar{\sigma}(t,T+\delta)dt + \gamma(t,T)L(t,T)d\tilde{B}(t)
\end{aligned}
$$

where $\gamma(t,T) = \frac{1+\delta L(t,T)}{\delta L(t,T)}\bar{\sigma}(t,T+\delta)\bar{\sigma}(t,T)$. Since what we want is a log-normal volatility stricture, we need to make the assumption that our volatility process, $\sigma(\cdot,\cdot)$ and $L(\cdot,\cdot)$, is dependent of each other in such a way that $\gamma(t,T)$ becomes deterministic.

To find the risk neutral probability measure for our LIBOR-model, we define the T-forward measure:

Definition 1 (T-forward measure)**.** The process $\tilde{B}^T(t)$ is called a T-forward Brownian motion, and it is given by:

$$\tilde{B}^T(t) = \tilde{B}(t) + \int_0^t \bar{\sigma}(s,T)ds$$

and $\tilde{B}^T(t)$ is a Brownian motion under the T-forward measure defined by:

$$Q^T(A) = E_Q\left[\mathbb{1}(A)Z^T(T)\right]$$

Where $Z^T(T)$ is given by:

$$Z^T(t) = \exp\left(-\int_0^t \bar{\sigma}(s,T)d\tilde{B}(t) - \frac{1}{2}\int_0^t \bar{\sigma}^2(s,T)ds\right)$$

where $\tilde{B}(t)$ is the Brownian motion under the measure we defined as Q.

We will now show that this measure also can be written in the way:

$$Q^T(A) = \frac{1}{P(0,T)} \cdot E_Q[\mathbb{1}(A)D(T)]$$

which means that $\frac{D(T)}{P(0,T)} = Z^T(T)$, where $D(t) = \exp\left(-\int_0^t r(s)ds\right) = \beta(t)^{-1}$.

First we use Itô's formula on the process $D(t)P(t,T)$, which gives us:

$$\begin{aligned} d(D(t)P(t,T)) &= D(t)P(t,T)r(t)dt - D(t)\bar{\sigma}(t,T)P(t,T)d\tilde{B}(t) \\ &\quad +P(t,T)(-r(t))D(t)dt \\ &= -D(t)\bar{\sigma}(t,T)P(t,T)d\tilde{B}(t) \end{aligned}$$

and this is a stochastic differential equation in $D(t)P(t,T)$, with solution:

$$D(t)P(t,T) = P(0,T)\exp\left(-\int_0^t \bar{\sigma}(s,T)d\tilde{B}(s) - \frac{1}{2}\int_0^t \bar{\sigma}^2(s,T)ds\right) \quad (2.18)$$

but this is the same as $P(0,T)Z^T(t)$. This means that

$$Z^T(T) = \frac{D(T)}{P(0,T)}P(T,T) = \frac{D(T)}{P(0,T)} \quad (2.19)$$

since $P(T,T) = 1$, which is what we wanted to show.

If we use the Brownian motion from the $T+\delta$-forward measure, $\tilde{B}^{T+\delta}(t)$, we will get this expression for $dL(t,T)$:

$$dL(t,T) = \gamma(t,T)L(t,T)d\tilde{B}^{T+\delta}(t) \quad (2.20)$$

Now, under the assumptions we have made, we have shown that the volatility structure of $L(t,T)$ is log-normal, and we have found a suitable probability measure, $Q^{T+\delta}$, under which $L(t,T)$ is a martingale, then the next step is to look at how to compute option prices for this interest rate model.

2.4 The European Call Option for the LIBOR Model

In this section we will see how to price options when the underlying is a forward rate modeled by the LIBOR model. The option we want to look at is the interest rate cap, which pays the difference between a fixed interest rate(cap-rate), and a variable interest rate, when the variable rate goes over the set cap-rate. This is to insure the holder against interest rates that goes higher then wanted. In mathematical terms, this can be described by $(L(t,T)-K)^+ = \max(L(t,T)-K,0)$

So what we want to compute is the value of a caplet that pays $(L(T,T)-K)^+$ at the time point $T+\delta$, where $K>0$ is a constant. This is equivalent to computing:

$$CapletValue_0 = CV_0 = E_Q\left[\exp\left(-\int_0^{T+\delta} r(s)ds\right)(L(T,T)-K)^+\right]$$

To compute this we will first show that the integral of a deterministic function with respect to a Brownian motion is normally distributed.

Theorem 2. If $f:\mathbb{R}\to\mathbb{R}$, is a deterministic function, with bounded second moments, $E[(\int_s^t f(u)dB(u))^2]<\infty$. Then $\int_s^t f(u)dB(u) \sim \mathcal{N}(0,\int_s^t f^2(u)du)$.

Proof. The characteristic function of a random variable Y is given by $\varphi_Y(\lambda) = E[\exp(i\lambda Y)]$, $\lambda\in\mathbb{R}$. Two random variables $Y,Z\in\mathbb{R}$, has the same distribution iff. $\varphi_Y(\lambda)=\varphi_Z(\lambda)$, for all $\lambda\in\mathbb{R}$. The characteristic function of a normal random variable $X\sim\mathcal{N}(\mu,\sigma^2)$ is given by $\varphi_X(\lambda)=\exp(i\mu\lambda-\frac{1}{2}\sigma^2\lambda^2)$

The characteristic function of $X_t=\int_s^t f(u)dB(u)$ is given by $\varphi_{X_t}(\lambda) = E[\exp(i\lambda X_t)]$. Then we can use the fact that X_t is an Itô-integral, and we can write it as the random variable:

$$X_t = \int_s^t f(u)dB(u) = \lim_{k\to\infty}\sum_{i=1}^{k-1} f(s_i^k)(B(s_{i+1}^k)-B(s_i^k))$$

where $s_1^k < s_2^k < \ldots. < s_n^k$ is partitions of $[s,t]$. Since the parts of the sum $f(s_i^k)(B(s_{i+1}^k)-B(s_i^k))$ are independent of each other, and normally distributed, $\mathcal{N}(0,f(s_i^k)(s_{i+1}^k-s_i^k))$, we can easily compute the expectation of each term like this:

$$E[\exp\left\{i\lambda f(s_i^k)(B(s_{i+1}^k) - B(s_i^k))\right\}] = \exp\left(-\frac{1}{2}\lambda^2 f^2(s_i^k)(s_{i+1}^k - s_i^k)\right)$$

Then we can compute $\varphi_{X_t}(\lambda)$.

$$\begin{aligned}
\varphi_{X_t}(\lambda) \quad &= E\left[\exp\left(i\lambda \int_s^t f(u)dB(u)\right)\right] \\
&= E\left[\exp\left(i\lambda \lim_{k\to\infty} \sum_{i=1}^{k-1} f(s_i^k)(B(s_{i+1}^k) - B(s_i^k))\right)\right] \\
&= E\left[\lim_{k\to\infty} \prod_{i=1}^{k-1} \exp\left(i\lambda f(s_i^k)(B(s_{i+1}^k) - B(s_i^k))\right)\right] \\
&\overset{(1)}{=} \lim_{k\to\infty} \prod_{i=1}^{k-1} E\left[\exp\left(i\lambda f(s_i^k)(B(s_{i+1}^k) - B(s_i^k))\right)\right] \\
&= \lim_{k\to\infty} \prod_{i=1}^{k-1} E\left[\exp\left(-\frac{1}{2}\lambda^2 f^2(s_i^k)(s_{i+1}^k - s_i^k)\right)\right] \\
&\overset{(2)}{=} E\left[\exp\left(-\frac{1}{2}\lambda^2 \lim_{k\to\infty} \sum_{i=1}^{k-1} f^2(s_i^k)(s_{i+1}^k - s_i^k)\right)\right] \\
&\overset{(3}{=} \exp\left(-\frac{1}{2}\lambda^2 \int_s^t f^2(u)du\right) \qquad (2.21)
\end{aligned}$$

(1) Here we use the dominated convergence theorem to take out the limit, then we need that $|\prod_{i=1}^{n-1} \exp\left(i\lambda f(s_i)(B(s_{i+1}) - B(s_i))\right)| \leq g$ for some integrable function g, but if f is real-valued, then $|\exp\left(i\lambda f(s_i)(B(s_{i+1}) - B(s_i))\right)| = 1$, so this is clear. When we have this, we can also take the product of expectations, because of independence.
(2) Here we take the limit and product inside again, in the same way as earlier, with dominated convergence theorem, and continuity of $\exp(x)$.
(3) Here we take the limit of the sum, and use that this is a Riemann integral, to get our result, then we see our last term is the characteristic function of a normal random variable, given by $\mathcal{N}(0, \int_s^t f^2(u)du)$, which means that X_t has the wanted distribution.

□

So, to compute CV_0, we first notice that $\exp(-\int_0^{T+\delta} r(s)ds){=}D(T+\delta)$, then we divide and multiply with $P(0,T+\delta)$, which gives us this:

$$\begin{aligned} CV_0 &= P(0,T+\delta)\frac{1}{P(0,T+\delta)}E_Q[D(T+\delta)(L(T,T)-K)^+] \\ &= P(0,T+\delta)E_{Q^{T+\delta}}[(L(T,T)-K)^+] \end{aligned} \tag{2.22}$$

To compute this, note that $L(T,T)$ is given by:

$$dL(t,T) = \gamma(t,T)L(t,T)d\tilde{B}^{T+\delta}(t)$$

where $\tilde{B}^{T+\delta}(t)$ is a Brownian motion under $Q^{T+\delta}$, then by Itô's formula we get:

$$L(t,T) = L(0,T)\exp\left(-\frac{1}{2}\int_0^t \gamma^2(s,T)ds + \int_0^t \gamma(s,T)d\tilde{B}^{T+\delta}(s)\right)$$

Since we have assumed $\gamma(s,T)$ to be deterministic, $\int_0^t \gamma(s,T)d\tilde{B}^{T+\delta}(s) \sim \mathcal{N}\left(0, \sqrt{\int_0^t \gamma^2(s,T)ds}\right)$.

If we then denote $\bar{\gamma}(t) = \sqrt{\int_0^t \gamma^2(s,T)ds}$, and we say $Y \sim \mathcal{N}(0,1)$, then we have:

$$L(T,T) \sim L(0,T)\exp\left(-\frac{1}{2}\bar{\gamma}^2(T) + \bar{\gamma}(T)\cdot Y\right)$$

So to compute the value for CV_0, we use the same argumentation which is used for the Black & Scholes pricing formula for call options.

First we denote $d_\pm = \frac{\ln(L(0,T)/K)\pm\frac{1}{2}\bar{\gamma}^2(T)}{\bar{\gamma}(T)}$, then we see that $L(T,T) > K$, when $Y > -d_-$. Using this, we can compute CV_0 like this:

$$
\begin{aligned}
E_{Q^{T+\delta}}[(L(T,T)-K)^+] &= E_{Q^{T+\delta}}[(L(T,T)-K)\cdot \mathbb{1}_{\{Y>-d_-\}}] \\
&= E\left[\left(L(0,T)\exp\left(-\frac{1}{2}\bar{\gamma}^2(T)+\bar{\gamma}(T)\cdot Y\right)-K\right)\mathbb{1}_{\{Y>-d_-\}}\right] \\
&\overset{(1)}{=} \int_{\mathbb{R}}\left(L(0,T)e^{-\frac{1}{2}\bar{\gamma}^2(T)+\bar{\gamma}(T)\cdot y}-K\right)\mathbb{1}_{\{y>-d_-\}}\frac{1}{\sqrt{2\pi}}\cdot e^{-y^2/2}dy \\
&\overset{(2)}{=} \int_{\mathbb{R}} L(0,T)e^{-\frac{1}{2}\bar{\gamma}^2(T)+\bar{\gamma}(T)\cdot y}\mathbb{1}_{\{y>-d_-\}}\frac{1}{\sqrt{2\pi}}\cdot e^{-y^2/2}dy - K\Phi(d_-) \\
&\overset{(3)}{=} N\cdot C\int_{-d_-}^{\infty} e^{\left(\frac{2\bar{\gamma}(T)\cdot y-y^2}{2}\right)}dy - K\Phi(d_-) \\
&\overset{(4)}{=} N\cdot C\int_{-d_-}^{\infty} e^{\left(\frac{-(y-\bar{\gamma}(T))^2+\bar{\gamma}(T)^2}{2}\right)}dy - K\Phi(d_-) \\
&\overset{(5)}{=} N\cdot L(0,T)\int_{-d_-}^{\infty} e^{\left(\frac{-(y-\bar{\gamma}(T))^2}{2}\right)}dy - K\Phi(d_-) \\
&\overset{(6)}{=} N\cdot L(0,T)\int_{-d_--\bar{\gamma}(T)}^{\infty} e^{\left(\frac{-x^2}{2}\right)dx-K\Phi(d_-)} \\
&\overset{(7)}{=} L(0,T)\Phi(d_+)-K\Phi(d_-)
\end{aligned} \tag{2.23}
$$

(1)Here we use that Y is normally distributed, and we use the definition of the expected value.
(2) Here we use the definition of the cumulative distribution function for a standard normal random variable Φ.
(3) Here we set everything into one exponential, and introduce the constants $N = 1/\sqrt{2\pi}$ and $C = L(0,T)\exp(-\frac{1}{2}\bar{\gamma}^2(T))$.
(4) Here we complete the square in the exponential.
(5) Here we remove everything independent of y, so we get back the constant $L(0,T)$.
(6) Here we make the change of variable $x = y - \bar{\gamma}(T)$.
(7) Here we again use the definition of the cumulative distribution function for a standard normal random variable Φ, and we get our final result.

Using (2.22) and (2.23), we get this formula for our caplet value:

$$CV_0 = P(0,T+\delta)\cdot[L(0,T)\Phi(d_+)-K\Phi(d_-)] \tag{2.24}$$

3 Theoretical Background

In this chapter we will go through theory for Lévy processes and jump diffusions, and present some results related to these processes. We will go through known results related to Itô-Lévy processes, like the Itô-Lévy formula, Girsanov's theorem and the maximum principle. We will also introduce an Esscher transform for Itô-Lévy processes, and show how to calculate characteristic functions for Itô-Lévy processes with deterministic integrands. We will also go through some more advanced methods in this thesis that might be unknown to the reader, in Section 3.7 we show the connection between arbitrage free prices and risk neutral measures, in Section 3.8 we will show how we can use Fourier transforms to compute arbitrage free prices, and in Section 3.9 we will introduce a duality method to find optimal portfolios when we want to maximize the expected utility from an investment.

3.1 Lévy processes

Lévy processes are a class of stochastic processes, which includes the Brownian motion, but also include more general types of processes, and are defined like this:

Definition 2. Lévy processes[6, P.68, Def. 3.1]
A Lévy process, $(L_t)_{t\geq 0}$, is a stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $\mathbb{R}^d$, where $L_0 = 0$, and it has these properties:

(i) It has independent increments, so for $t_0 \leq t_1 \leq t_2, \leq t_3$, we have that the random variables $L_{t_0}, L_{t_1} - L_{t_0}, L_{t_2} - L_{t_1}$ are independent.

(ii) It has stationary increments, so the law of $L_{t+h} - L_t$ is only dependent of h and not of t.

(iii)It has stochastic continuity, so $\forall \epsilon > 0$, $\lim_{h\to 0} \mathbb{P}(|L_{t+h} - L_t| \geq \epsilon) = 0$.

The third condition does not mean the path of a Lévy process is necessary continuous, but we need that for a time point t that the probability for a jump to happen at that point of time is equal to zero, this implies we can't have jumps at given deterministic times.

There is also shown that any Lévy process has a unique cádlág(right continuous with left limits) version, so we assume our Lévy processes has this property. The meaning of this, is that if we have a jump at time t, then L_t is the value after the jump, and we denote by $L_{t^-} = \lim_{h \to t^-} L_h$ the value just before the jump.

As we see, the Brownian motion satisfies the definition of a Lévy process, but we also have other processes which satisfies this. Another example which satisfies the definition of a Lévy process is the Poisson process $N(t)$ with intensity λ and probability distribution equal to:

$$P(N(t) = n) = \frac{(\lambda t)^n}{n!} e^{-\lambda t} \tag{3.1}$$

We also have the more general compound Poisson process, $Y(t)$, which sums up a number of i.i.d. jumps. Here the number of jumps are given by the Poisson process $N(t)$, and the number of jumps are independent of the jump size.

$$Y(t) = \sum_{i=1}^{N(t)} X(i) \tag{3.2}$$

Here $X(i)$ is a sequence of i.i.d random variables, which gives us the jump sizes. We can only write this is we have a finite Lévy measure ν, if it is not finite we can have infinitely "small" jumps.

In general we can decompose a Lévy process into four different terms, a drift part, a Brownian motion part, a ”small” jump part, and a ”large” jump part.

Theorem 3. Itô-Lévy Decomposition[6, P. 79, Prop. 3.7]
If L_t is a Lévy process, then it can be decomposed like this:

$$L_t = \alpha t + \sigma B(t) + \int_{|z|<\mathbb{R}} z\tilde{N}(t, dz) + \int_{|z|\geq\mathbb{R}} zN(t, dz) \tag{3.3}$$

here $\alpha, \sigma \in \mathbb{R}$ and $R \in [0, \infty]$, and

$$\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)dt \tag{3.4}$$

is the compensated Poisson random measure, $\nu(U) = E[N(1, U)]$, is the Lévy measure, which is the expected number of jumps which ends in the set U, in the time interval $(0, 1]$. We also need our Brownian motion $B(t)$ to be independent of $\tilde{N}(dt, dz)$. We call (α, σ, ν) for the characteristic triplet of our Lévy process L_t, and these uniquely determines the Lévy process.

The constant R can be chosen as small as we want, but in some cases we can expect infinitely many small jumps, and therefore we can have that $\int_{|z|<R} |z|\nu(dz) = \infty$, so we need to compensate our Poisson random measure, $N(t, dz)$ around 0.

Using the fact that $N(t, U)$ has independent increments, it is easy to show that our compensated Poisson random measure $\tilde{N}(t, u)$ is a martingale.

$$\begin{aligned} E[\tilde{N}(t,U)|\mathcal{F}_s] &= E[N(t,U) - \nu(U)t|\mathcal{F}_s] = E[N(t,U) - N(s,U)|\mathcal{F}_s] + \\ & E[N(s,U)|\mathcal{F}_s] - \nu(U)t = E[N(t,U) - N(s,U)] + N(s,U) - \\ & \nu(U)t = \nu(U)(t-s) + N(s,U) - \nu(U)t = \tilde{N}(s,U) \end{aligned} \tag{3.5}$$

Theorem 4. The Lévy-Khintchine formula[6, P.83, Th. 3.1]
If L_t is a Lévy process with Lévy measure ν. Then $\int_{\mathbb{R}} \min(1, z^2)\nu(dz) < \infty$ and

$$E[e^{iuL_t}] = e^{t\psi(u)}, \ u \in \mathbb{R} \tag{3.6}$$

where

$$\psi(u) = \frac{1}{2}\sigma^2 u^2 + i\alpha u + \int_{|z|<R} \{e^{iuz} - 1 - iuz\}\nu(dz) + \int_{|z|\geq R} \{e^{iuz} - 1\}\nu(dz) \tag{3.7}$$

3.2 Itô-Lévy processes

Now we have defined Lévy processes L_t, but we are interested in more general processes, what we want is to integrate a process with respect to a Lévy process, like this:

$$X(t) = X(0) + \int_0^t \alpha(s,\omega)ds + \int_0^t \beta(s,\omega)dB(s) + \int_0^t \int_{\mathbb{R}} \gamma(s,z,\omega)\bar{N}(ds,dz) \tag{3.8}$$

where $\bar{N}(dt, dz)$, is given by:

$$\bar{N}(dt,dz) = \begin{cases} N(dt,dz) - \nu(dz)dt & \text{if } |z| < R \\ N(dt,dz) & \text{otherwise} \end{cases} \tag{3.9}$$

we will also use the shorthand notation:

$$dX(t) = \alpha(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}} \gamma(t,z)\bar{N}(dt,dz) \tag{3.10}$$

For these integrals to exist we need certain conditions on our integrands for them to be well defined. In general we want our integrands to be predictable, which means they are measurable with respect to the σ-algebra $\mathcal{F}_{t-}$ for each time point t. We call processes on the form of (3.8) for Itô-Lévy processes.

Since $\tilde{N}(t,U)$ is a martingale, it is natural to assume that integration with respect to this, will give us a martingale. If we define a process $M(t)$ like this:

$$M(t) = \int_0^t \int_{\mathbb{R}} \gamma(s,z)\tilde{N}(ds,dz); \ 0 \leq t \leq T \tag{3.11}$$

then $M(t)$ is a martingale if $E\left[\int_0^T \int_{\mathbb{R}} \gamma^2(t,z)\nu(dz)dt\right] < \infty$, and $M(t)$ is a *local* martingale if $\int_0^T \int_{\mathbb{R}} \gamma^2(t,z)\nu(dz)dt < \infty$ a.s.

Note that if $\gamma(t,z)$ is deterministic, then

$$E\left[\int_0^T \int_{\mathbb{R}} \gamma^2(t,z)\nu(dz)dt\right] = \int_0^T \int_{\mathbb{R}} \gamma^2(t,z)\nu(dz)dt$$

so in this case, if $M(t)$ is a local martingale, then $M(t)$ is also a martingale. In this thesis we will generally end up with deterministic integrands, so for us *local* martingales and martingales will be the same.

The next thing we want to look at is processes of the form $f(t,X(t))$, where $X(t)$ is given by (3.8). Then we need, as in the Brownian motion case, an Itô formula, but now for Itô-Lévy processes.

Theorem 5. One-dimensional Itô formula[1, P.7, Th. 1.14 and references herein]
If we have an Itô-Lévy process, $X(t)$, given by $dX(t) = \alpha(t)dt + \beta(t)dB(t) + \int_{\mathbb{R}} \gamma(t,z)\bar{N}(dt,dz)$, and we have a C^2 function $f : \mathbb{R}^2 \to \mathbb{R}$, then the process $Y(t) := f(t,X(t))$ is again an Itô-Lévy process, and it is given by:

$$
\begin{aligned}
dY(t) &= \frac{\partial f}{\partial t}(t, X(t))dt + \frac{\partial f}{\partial x}(t, X(t))dX^c(t) + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(t, X(t)) \cdot \beta^2(t)dt \\
&\quad + \int_{\mathbb{R}} \left[f(t, X(t^-) + \gamma(t,z)) - f(t, X(t^-))\right] N(dt, dz) \\
&= \frac{\partial f}{\partial t}(t, X(t))dt + \frac{\partial f}{\partial x}(t, X(t))[\alpha(t)dt + \beta(t)dB(t)] \\
&\quad + \frac{1}{2}\beta^2(t)\frac{\partial^2 f}{\partial x^2}(t, X(t))dt \\
&\quad + \int_{|z|<R} \Big[f(t, X(t^-) + \gamma(t,z)) - f(t, X(t^-)) \\
&\quad - \frac{\partial f}{\partial x}(t, X(t^-))\gamma(t,z) \Big] \nu(dz)dt \\
&\quad + \int_{\mathbb{R}} \left[f(t, X(t^-) + \gamma(t,z)) - f(t, X(t^-))\right] \bar{N}(dt, dz)
\end{aligned} \tag{3.12}
$$

Here $X^c(t)$ is the continuous part of $X(t)$, what we get if we remove the jumps from $X(t)$.

$$
dX^c(t) = \left(\alpha(t) - \int_{|z|<R} \gamma(t,z)v(dz)\right) dt + \beta(t)dB(t)
$$

$X(t^-) = \lim_{y \to t^-} X(y)$, is the left limit of $X(t)$, so if we have a jump at time t, $X(t^-)$ is the value just before the jump.

Proof. We will not give a full proof, but a sketch of what happens, and only for the discontinuous part, since the continuous part should be known.

So what we want to look at, is a process of the form $f(X(t))$, where $X(t) = \int_0^t \int_{\mathbb{R}} \gamma(s,z)N(ds,dz)$. At time t we will have a number of jumps(in theory we will have at most 1 jump, since we model the number of jumps by a Poison process). If the Lévy process jumps, we will go from $X(t^-)$, to $X(t^-) + \gamma(t,z)$, if it don't jump, nothing happens.

This means that the difference in $f(X(t))$ at the time of a jump, will be $f(X(t^-) + \gamma(t,z)) - f(X(t^-))$, and $df(X(t))$ can be written as:

$$
df(X(t)) = \int_{\mathbb{R}} f(X(t^-) + \gamma(t,z)) - f(X(t^-))N(dt, dz) \tag{3.13}
$$

To get the wanted result, we add and subtract $\int_{|z|<R} f(X(t^-)+\gamma(t,z)) - f(X(t^-))\nu(dz)$, so we can use $\bar{N}(dt,dz)$ instead of $N(dt,dz)$. □

Now we have a way to compute $f(t,X(t))$, and the next thing we will present is the Itô-Lévy Isometry, which is a way to compute variances for Itô-Lévy processes.

Theorem 6. The Itô-Lévy Isometry
If $X(t)$ is an Itó-Levy process given like this:

$$dX(t) = \sigma(t)dB(t) + \int_{\mathbb{R}} \gamma(t,z)\tilde{N}(dt,dz);\ 0 \le t \le T \tag{3.14}$$

and $X(0) = 0$, then:

$$E[X^2(t)] = E\left[\int_0^t \sigma^2(s)ds + \int_0^t \int_{\mathbb{R}} \gamma^2(s,z)\nu(dz)ds\right] \tag{3.15}$$

If our right-hand side is finite.

Proof. If we use Itô's formula, with $f(t,x) = x^2$ on our process $X(t)$, then we get:

$$\begin{aligned}
d(X^2(t)) &= 2X(t)\sigma(t)dB(t) + \sigma^2(t)dt + \int_{\mathbb{R}} [(X(t^-)+\gamma(t,z))^2 - X^2(t) \\
&\quad -2\gamma(t,z)X(t)]\nu(dz)dt + \int_{\mathbb{R}} [(X(t^-)+\gamma(t,z))^2 - X^2(t)]\tilde{N}(dt,dz) \\
&= 2X(t)\sigma(t)dB(t) + \sigma^2(t)dt + \int_{\mathbb{R}} \gamma^2(t,z)\nu(dz)dt \\
&\quad + \int_{\mathbb{R}} [2X(t^-)\gamma(t,z) + \gamma^2(t,z)]\tilde{N}(dt,dz) \\
&= 2X(t)\sigma(t)dB(t) + \sigma^2(t)dt + \int_{\mathbb{R}} \gamma^2(t,z)\nu(dz)dt \\
&\quad + \int_{\mathbb{R}} [2X(t^-)\gamma(t,z) + \gamma^2(t,z)]\tilde{N}(dt,dz)
\end{aligned} \tag{3.16}$$

which means:

$$\begin{aligned}
X^2(t) &= \int_0^t 2X(s)\sigma(s)dB(s) + \int_0^t \sigma^2(s)ds + \int_0^t \int_{\mathbb{R}} \gamma^2(s,z)\nu(dz)ds \\
&\quad + \int_0^t \int_{\mathbb{R}} [2X(s^-)\gamma(s,z) + \gamma^2(s,z)]\tilde{N}(dt,dz)
\end{aligned} \tag{3.17}$$

Then if our integrals with respect to our Brownian motion and our compensated Poisson random measure are properly bounded, they will be martingales, and we get that:

$$E\left[\int_0^t 2X(s)\sigma(s)dB(s) + \int_0^t \int_{\mathbb{R}} [2X(s^-)\gamma(s,z) + \gamma^2(s,z)]\tilde{N}(ds,dz)\right] = 0 \tag{3.18}$$

and we can conclude that:

$$E[X^2(t)] = E\left[\int_0^t \sigma^2(s)ds + \int_0^t \int_{\mathbb{R}} \gamma^2(s,z)\nu(dz)ds\right] \tag{3.19}$$

□

3.3 Girsanov's Theorem

In this section we will state the Girsanov theorem for Itô-Lévy processes, which is used to change the probability measure so our Itô-Lévy process becomes a martingale under the new measure. The reason for why we want such a measure, is that under this measure we can take the expectation of the discounted option price, where the underlying is our Itô-Lévy process, and the value we get will be an arbitrage free price for the option.

This new measure will need to be equivalent to the old one, and we say that two probability measures Q and P are equivalent if they have the same zero sets, which is the same as $P \ll Q$ and $Q \ll P$. What we have is the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, so we want our probability measure Q on $\mathcal{F}_T$ to be equivalent to $P|\mathcal{F}_T$. By the Radon-Nikodym derivative this is the same as $dQ(\omega) = Z(T)dP(\omega)$ and $dP(\omega) = Z^{-1}(T)dQ(\omega)$, for some $\mathcal{F}_T$-measurable random variable $Z(T)$. And for all $t \in [0,T]$, we have that $\frac{dQ|\mathcal{F}_t}{dP|\mathcal{F}_t} = Z(t)$, and we need $Z(t) > 0$ for $0 \leq t \leq T$.

To see how we can change a probability measure, such that we get a martingale under the new measure, we will use Bayes' rule. Bayes' rule says that if we have two probability measures Q and $P|\mathcal{F}_T$, such that $dQ(\omega) = Z(T)dP(\omega)$, and a random variable X, where $E_Q[|X|] < \infty$, then:

$$E_Q[X|\mathcal{F}_t] = \frac{E_P[Z(T)X|\mathcal{F}_t]}{E_P[Z(T)|\mathcal{F}_t]} \tag{3.20}$$

Having this, we can look at a process $X(t)$, and we use that the process $Z(t)$ defined earlier is a $P-martingale$, then if $Z(t)X(t)$ is a $P-martingale$, we get that $X(t)$ is a $Q-martingale$. To show this we set $s \geq t$, and we get:

$$\begin{aligned} E_Q[X(s)|\mathcal{F}_t] &= \frac{E_P[Z(T)X(s)|\mathcal{F}_t]}{E_P[Z(T)|\mathcal{F}_t]} \\ &= \frac{E_P[E_P[Z(T)X(s)|\mathcal{F}_s]|\mathcal{F}_t]}{Z(t)} \\ &= \frac{E_P[X(s)Z(s)|\mathcal{F}_t]}{Z(t)} \\ &= \frac{X(t)Z(t)}{Z(t)} \\ &= X(t) \end{aligned} \tag{3.21}$$

Here we have used conditional expectation, and that $X(s)$ is $\mathcal{F}_s$-measurable, in the same way we can show that if $Z(t)X(t)$ is a local $P-martingale$, then $X(t)$ is a local $Q-martingale$.

So to find an equivalent martingale measure, we would like to find a process $Z(t)$, which is a martingale, and such that $Z(t)X(t)$ is a martingale(local martingale). To find this measure, we need to compute $Z(t)X(t)$, and to compute this we need a product rule for Itô-Lévy processes. If we have two processes $Y(t)$ and $X(t)$, defined by:

$$dX(t) = a(t)dt + b(t)dB(t) + \int_{\mathbb{R}} g(t,z)\tilde{N}(dt,dz) \tag{3.22}$$

and

$$dY(t) = u(t)dt + v(t)dB(t) + \int_{\mathbb{R}} f(t,z)\tilde{N}(dt,dz) \tag{3.23}$$

then we can use the function $f(t, x, y) = x \cdot y$, and a multidimensional Itô-Lévy formula[1, P. 8, Th 1.16] on $f(t, X(t), Y(t))$ to get:

$$
\begin{aligned}
d(X(t)Y(t)) &= Y(t)(a(t)dt + b(t)dB(t)) + X(t)(u(t)dt + v(t)dB(t)) \\
&\quad + b(t)u(t)dt + \int_{\mathbb{R}} [(X(t^-) + g(t,z))(Y(t^-) + f(t,z)) - X(t^-)Y(t^-) \\
&\quad - g(t,z)Y(t^-) - f(t,z)X(t^-)]\nu(dz)dt \\
&\quad + \int_{\mathbb{R}} [(X(t^-) + g(t,z))(Y(t^-) + f(t,z)) - X(t^-)Y(t^-)]\tilde{N}(dt,dz) \\
&= Y(t)(a(t)dt + b(t)dB(t)) + X(t)(u(t)dt + v(t)dB(t)) \\
&\quad + b(t)u(t)dt + \int_{\mathbb{R}} g(t,z)f(t,z)\nu(dz)dt \\
&\quad + \int_{\mathbb{R}} [X(t^-)f(t,z) + Y(t^-)g(t,z) + g(t,z)f(t,z)]\tilde{N}(dt,dz)
\end{aligned}
\tag{3.24}
$$

Using this we can state our Girsanov theorem, which will be shown in one dimension, since that is what we are interested in, the n-dimensional case is similar:

Theorem 7. Girsanov Theorem for Itô-Lévy Processes
If we have an Itô-Lévy process $X(t)$, of the form:

$$dX(t) = \alpha(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}} \gamma(t,z)\tilde{N}(dt,dz) \tag{3.25}$$

and we have predictable processes $u(t,\omega) = u(t)$ and $\theta(t,z,\omega) = \theta(t,z)$, which satisfies:

$$\sigma(t)u(t) + \int_{\mathbb{R}} \gamma(t,z)\theta(t,z)\nu(dz) = \alpha(t) \tag{3.26}$$

and the process

$$
\begin{aligned}
Z(t) :&= \exp\Bigg(-\int_0^t u(s)dB(s) - \frac{1}{2}\int_0^t u^2(s)ds \\
&\quad + \int_0^t \int_{\mathbb{R}} \ln(1-\theta(s,z))\tilde{N}(ds,dz) \\
&\quad + \int_0^t \int_{\mathbb{R}} (\ln(1-\theta(s,z)) + \theta(s,z))\nu(dz)ds\Bigg);\ 0 \le t \le T
\end{aligned}
\tag{3.27}
$$

is well defined and satisfies

$$E[Z(T)] = 1 \tag{3.28}$$

If we then define the probability measure Q by $dQ(\omega) = Z(T)dP(\omega)$. Then $X(t)$ is a martingale(local martingale) with respect to Q.

Proof. From earlier, we know that this is true if $Z(t)$ and $Z(t)X(t)$ are martingales(local martingales), so we will show this. If we use Itô's formula on $Z(t)$, we get:

$$dZ(t) = -Z(t)u(t)dB(t) - Z(t^-)\int_{\mathbb{R}} \theta(t,z)\tilde{N}(dt,dz) \tag{3.29}$$

and if

$$E\left[\int_0^T (Z(t)u(t))^2 dt + \int_0^T \int_{\mathbb{R}} (Z(t)\theta(t,z))^2 \nu(dz)dt\right] < \infty \tag{3.30}$$

then $Z(t)$ is a martingale, and $E[Z(T)] = Z(0) = 1$. If we use (3.24) on our product $Z(t)X(t)$, we get:

$$\begin{aligned} d(Z(t)X(t)) &= Z(t)(\alpha(t)dt + \sigma(t)dB(t)) + Z(t^-)\int_{\mathbb{R}} \gamma(t,z)\tilde{N}(dt,dz) \\ &\quad -X(t)Z(t)u(t)dB(t) - X(t^-)Z(t^-)\int_{\mathbb{R}} \theta(t,z)\tilde{N}(dt,dz) \\ &\quad -\sigma(t)Z(t)u(t)dt - Z(t^-)\int_{\mathbb{R}} \gamma(t,z)\theta(t,z)\tilde{N}(dt,dz) \\ &\quad -Z(t)\int_{\mathbb{R}} \gamma(t,z)\theta(t,z)\nu(dz)dt \end{aligned} \tag{3.31}$$

And we see the only terms that are not martingales are:

$$Z(t)\alpha(t) - Z(t)\sigma(t)u(t) - Z(t)\int_{\mathbb{R}} \gamma(t,z)\theta(t,z)\nu(dz) \tag{3.32}$$

and these are the same terms as we had in (3.26), so from our assumption this is equal to 0, and we are left with only martingale terms, so $Z(t)X(t)$ is a martingale(local martingale) under the right integrability conditions.

□

There are two things we have to keep in mind with this Girsanov theorem which are different from the one we get when we only have Brownian motions. First of all, we have two functions we can change, $u(t)$ and $\theta(t,z)$, so we get several ways to change our measure, we can also change $\theta(t,z)$ for each jump size, so in general we won't have a complete market when we work with Lévy processes. We also have that while our Brownian motion will turn into a Brownian motion under our new measure, the Poisson random measure $N(dt,dz)$ will not necessary be a Poisson random measure under our new measure, so Itô's formula will not work.

For more general definitions of the Girsanov theorem, and more details on the proof, see [1, Ch. 1.4] and references herein.

3.4 Esscher Transform

Now we have shown how to change our probability measure with the Girsanov theorem, but for Lévy processes we have another way to define our Radon-Nikodym derivative, and that is the Esscher transform. The idea here is that if we have a Lévy process $L_t = \alpha t + \sigma B(t) + \int_{|z|<\mathbb{R}} z\tilde{N}(t,dz) + \int_{|z|\geq\mathbb{R}} zN(t,dz)$, and we take the exponential of this, $\exp(L_t)$, then we can define a Radon-Nikodym derivative like this:

$$\frac{dQ|\mathcal{F}_t}{dP|\mathcal{F}_t} = Z(t) = \frac{\exp(\theta L_t)}{E[\exp(\theta L_t)]} \tag{3.33}$$

We can then choose θ such that

$$E_Q[\exp(L_t)|\mathcal{F}_s] = \exp(L_s);\ \ s \leq t \tag{3.34}$$

This measure is called the compound return Esscher martingale measure. If we define $S(t) := \exp(L_t)$, then we can find a process $X(t)$ which satisfies $dS(t) = S(t^-)dX(t)$, and we can define a new Radon-Nikodym derivative by:

$$\frac{dQ|\mathcal{F}_t}{dP|\mathcal{F}_t} = Z(t) = \frac{\exp(\theta X(t))}{E[\exp(\theta X(t))]} \tag{3.35}$$

and also under this measure change, we can find a value of θ, such that $S(t)$ is a martingale under this change of measure. This is called the simple return Esscher transformed martingale measure, and it is known to be the same as the minimal entropy martingale measure, which we will describe later.

Since what we are interested in are Itô-Lévy processes, we will get a time dependence in our processes, so we will need a θ which is dependent of time. So if we have a process of the form $S(t) = \exp(L(t))$, where $L(t)$ is given by:

$$dL(t) = \alpha(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}} \gamma(t,z)\tilde{N}(dt,dz); \ 0 \leq t \leq T \tag{3.36}$$

Then we will define a Radon-Nikodym derivative by:

$$\frac{dQ|\mathcal{F}_t}{dP|\mathcal{F}_t} = Z(t) = \frac{\exp(\int_0^t \theta(s)dL(s))}{E[\exp(\int_0^t \theta(s)dL(s))]} \tag{3.37}$$

and if we have a process $X(t)$, which solves $dS(t) = S(t^-)dX(t)$, we can define a Radon-Nikodym derivative by:

$$\frac{dQ|\mathcal{F}_t}{dP|\mathcal{F}_t} = Z(t) = \frac{\exp(\int_0^t \theta(s)dX(s))}{E[\exp(\int_0^t \theta(s)dX(s))]} \tag{3.38}$$

and we can also find functions $\theta(s)$, which makes these measures martingale measures.

Later we will see how to find a $\theta(s)$ that solves this problem when we do this for our HJM model.

For more info on the Esscher transform, see [6, Ch. 9.5].

3.5 Characteristic Functions

In this section we will find the characteristic function of the discontinuous part of an Itô-Lévy process. The reason for why we want to compute this, is because in the Esscher transform we need to compute $E[\exp(X(t))]$, where $X(t)$ is an Itô-Lévy process, and this closely resembles the characteristic function.

So in this section we have a process of the form

$$M(t) = \int_0^t \int_{\mathbb{R}} \gamma(s,z)\tilde{N}(ds,dz) \tag{3.39}$$

and we want to compute the characteristic function of this. What we find is that the characteristic function of this is:

$$E[\exp(iuM(t))] = \exp\left\{\int_0^t \int_{\mathbb{R}} \left(e^{iu\gamma(s,z)} - 1 - iu\gamma(s,z)\right)\nu(dz)ds\right\} \tag{3.40}$$

and for this to hold we will need certain integrability conditions on γ, and we need γ to be deterministic. To show this we will separate our computation into three parts.

1. We will first assume that our function γ is independent of time, and we assume our Lévy measure ν is finite, and this is done so we can write our process as a sum of i.i.d. jumps, and we can compute the characteristic function of each jump.
2. Then we want to see what we get when the Lévy measure is infinite, and we do this by separating the "small" and the "large" jumps by a sequence of decreasing intervals, and we see what happens in the limit.
3. At last we introduce time again, and this is done by separating our time interval into a sequence of time points, and we look at our time integral as the limit of the sum of elementary functions.

So we first look at processes on the form $G(t) = \int_{\mathbb{R}} \gamma(z)N(t,dz)$, where γ is a deterministic function.

First of all, recall that if we have a finite Lévy measure ν, we have this equality:

$$\int_{\mathbb{R}} zN(t,dz) = \sum_{i=1}^{N(t)} Y(i) \tag{3.41}$$

Where $N(t)$ is the number of jumps up to time t, and $Y(i)$ is the size of jump number i, where $1 \leq i \leq N(t)$.

Since it is easier to work with sums then integrals, we would like a similar way to represent our process $G(t)$. In the process $G(t)$, we take a jump, $Y(i)$, and maps it into $\gamma(Y(i))$, so if we have a finite Lévy measure, we can write this:

$$G(t) = \int_{\mathbb{R}} \gamma(z)N(t,dz) = \sum_{i=1}^{N(t)} \gamma(Y(i)) \tag{3.42}$$

Now we look at the characteristic function of the first projected jump

$$E[e^{iu\gamma(Y(1))}] = \int_{\mathbb{R}} e^{iu\gamma(y)}\mu(dy) \tag{3.43}$$

where μ is the probability density of the jumps. Sine our jumps are independent and equally distributed we can calculate our characteristic function like this:

$$\begin{aligned}
E\left[e^{iu\int_{\mathbb{R}}\gamma(z)N(t,dz)}\right] &\overset{(1)}{=} E\left[E\left[e^{iu\sum_{i=1}^{N(t)}\gamma(Y(i))}|N(t)\right]\right] \\
&\overset{(2)}{=} \sum_{n=0}^{\infty} E\left[\prod_{i=1}^{N(t)} e^{iu\gamma(Y(i))}|N(t)=n\right] P(N(t)=n) \\
&\overset{(3)}{=} \sum_{n=0}^{\infty}\prod_{i=1}^{n} E\left[e^{iu\gamma(Y(i))}\right] P(N(t)=n) \\
&\overset{(4)}{=} \sum_{n=0}^{\infty}\left(\int_{\mathbb{R}} e^{iu\gamma(y)}\mu(dy)\right)^n P(N(t)=n) \\
&\overset{(5)}{=} \sum_{n=0}^{\infty}\left(\int_{\mathbb{R}} e^{iu\gamma(y)}\mu(dy)\right)^n e^{-\lambda t}\frac{(\lambda t)^n}{n!} \\
&\overset{(6)}{=} \exp\left\{\lambda t\left(\int_{\mathbb{R}} e^{iu\gamma(y)}\mu(dy)-1\right)\right\} \\
&\overset{(7)}{=} \exp\left\{t\int_{\mathbb{R}}\left(e^{iu\gamma(y)}-1\right)\nu(dy)\right\} \qquad (3.44)
\end{aligned}$$

(1) Here we write the integral as a sum, and we use that for two random variables Y, X, we have that $E[X]=E[E[X|Y]]$.
(2) Here we use that $e^{\sum_i X_i}=\prod_i e^{X_i}$, and that $E[X|Y]$ is a random variable with respect to Y, so $E[E[X|Y]]=\sum_y E[X|Y=y]P(Y=y)$.
(3) Here we use that we have conditioned on $N(t)=n$, and we take out the product, which can be done since the jumps are independent.
(4) Here we have computed the characteristic function of each jump, and we have used that the jumps are equally distributed.
(5) Here we have used that the number of jumps are Poisson distributed with parameter λt.
(6) Here we have first rewritten the sum, and then we have used the definition of the exponential formula $\exp(x)=\sum_{n=0}^{\infty}\frac{x^n}{n!}$.
(7) Here we have moved 1 into the integral, which we can do since μ is a probability distribution, and $\int_{\mathbb{R}}\mu(dy)=1$ and we have used that $\nu(dy)=\lambda\mu(dy)$.

So if we have a finite Lévy measure ν, then we can write our Itô-Lévy process as a sum of independent jumps, and if γ is deterministic, we get this characteristic function:

$$E\left[e^{iu\int_{\mathbb{R}}\gamma(z)\tilde{N}(t,dz)}\right] = \exp\left\{t\int_{\mathbb{R}}\left(e^{iu\gamma(y)} - 1 - iu\gamma(z)\right)\nu(dy)\right\} \tag{3.45}$$

The next thing we will do is to sketch a proof for what happens with infinite Lévy measures, for a more detailed proof for Lévy processes, see [6, Th. 3.1].

The problem with infinite Lévy measures is what happens around 0, since we can have infinitely many small jumps. To give an idea of how to work around this problem, we will bound our process away from zero, and see what happens in the limit. So if we define a sequence $\{N_k\}_{0\leq k\leq\infty}$, where $N_k \overset{k\to\infty}{\to} 0$, then the process $M_k(t)$:

$$M_k(t) = \int_{|z|>N_k}\gamma(z)\tilde{N}(t,dz) \tag{3.46}$$

will not include the small jumps. Since we have removed the small jumps, we can write $M_k(t)$ as a sum of jumps, $M_k(t) = \sum_{i=1,Y(i)>N_k}^{N(t)}\gamma(Y(i))$, and our previous result will hold. Now we see that the processes $M_k(t)$ are $L^2(P)$-martingales under some integrability conditions, and they will converge to a $L^2(P)$-martingale, this means they converge in distribution and then the characteristic function will converge. So our assumption is that we can write the characteristic function for our small jumps like this

$$E\left[e^{iu\int_{|z|<R}\gamma(z)\tilde{N}(t,dz)}\right] = \exp\left\{t\int_{|z|<R}\left(e^{iu\gamma(z)} - 1 - iu\gamma(z)\right)\nu(dz)\right\} \tag{3.47}$$

for some $R > 0$, and for this to be true we need our right-hand side to be finite. To find conditions so our right-hand side will be finite, we look at the Taylor polynomial of e^x around 0, and using this we get that $e^x \approx 1+x+\frac{e^c}{2}x^2$, for some $c \in [0,x]$. According to this $e^{iu\gamma(z)} - 1 - iu\gamma(z) \approx C\gamma^2(z)$, and we get that $C < \infty$ if $\gamma(z) < \infty$ for $z < R$, so we get this inequality

$$\int_{|z|<R}\gamma^2(z)\nu(dz) < \infty \tag{3.48}$$

need to hold for our right-hand side to be finite.

Now we have found a characteristic function when we map the jump size from $Y(i) \to \gamma(Y(i))$, but we are also interested in time dependent functions, so we want to look at the characteristic function of $\int_0^t\int_{\mathbb{R}}\gamma(s,z)\tilde{N}(ds,dz)$. To compute this, we will partition the interval $[0,t]$ by sequences $\{t_k^K\}_{1\leq k\leq K}$, where $0 = t_1^K < t_2^K ... t_{K-1}^K < t_K^K = T$, and $\lim_{K\to\infty} \sup_k |\Delta t_k^K| \to 0$, where $\Delta t_k^K = t_{k+1}^K - t_k^K$, this means the length of the largest interval goes to 0 when $K \to \infty$. Then we can compute the characteristic function like this:

$$\begin{aligned}
E\left[e^{iu\int_0^t \int_{\mathbb{R}} \gamma(s,z)\tilde{N}(ds,dz)}\right] &\overset{(1)}{=} E\left[e^{iu \lim_{K\to\infty} \sum_{k=1}^{K-1} \int_{\mathbb{R}} \gamma(t_k^K,z)\tilde{N}(\Delta t_k^K,dz)}\right] \\
&\overset{(2)}{=} E\left[\lim_{K\to\infty} \prod_{k=1}^{K-1} e^{iu\int_{\mathbb{R}} \gamma(t_k^K,z)\tilde{N}(\Delta t_k^K,dz)}\right] \\
&\overset{(3)}{=} \lim_{K\to\infty} \prod_{k=1}^{K-1} E\left[e^{iu\int_{\mathbb{R}} \gamma(t_k^K,z)\tilde{N}(\Delta t_k^K,dz)}\right] \\
&\overset{(4)}{=} \lim_{K\to\infty} \prod_{k=1}^{K-1} \exp\left\{\Delta t_k^K \int_{\mathbb{R}} \left(e^{iu\gamma(t_k^K,z)} - 1 - iu\gamma(t_k^K,z)\right)\nu(dz)\right\} \\
&\overset{(5)}{=} \exp\left\{\lim_{K\to\infty} \sum_{k=1}^{K-1} \Delta t_k^K \int_{\mathbb{R}} \left(e^{iu\gamma(t_k^K,z)} - 1 - iu\gamma(t_k^K,z)\right)\nu(dz)\right\} \\
&\overset{(6)}{=} \exp\left\{\int_0^t \int_{\mathbb{R}} \left(e^{iu\gamma(s,z)} - 1 - iu\gamma(s,y)\right)\nu(dz)ds\right\} \qquad (3.49)
\end{aligned}$$

(1) First we write our integral as a limit of the sum of integrals of elementary functions.
(2) Then we use that $\exp(\cdot)$ is continuous to take out the limit, and we use that $e^{\sum_i X_i} = \prod_i e^{X_i}$.
(3) Then we use dominated convergence to take the limit outside the expectation, this means we need functions that satisfy the conditions for the dominated convergence theorem.
(4) Then we use that for a given t_k^K we know the characteristic function of this from (3.44).
(5) Here we go back and use the continuity of $\exp(x)$ to take the limit in again.
(6) Finally we use that this is the limit of a sum of integrated simple functions, so the limit of the sums converge to an integral again, as wanted.

3.6 The Maximum Principle

The maximum principle is a way to choose control variables to maximize an expected value. A typical problem is an investment problem with several risky assets $\{S_i(t)\}_{0\leq i\leq I}$, then the maximum principle could be used to find out how much you should invest in each stock to maximize your expected gains.

The setup for the maximum principle is that you have a jump diffusion $X^u(t)$ which is dependent of some control $u(t)$, given like this:

$$\begin{aligned} dX^u(t) &= b(t, X(t), u(t))dt + \sigma(t, X(t), u(t))dB(t) \\ &\quad + \int_{\mathbb{R}} \gamma(t, X(t^-), u(t^-), z)\bar{N}(dt, dz); \; X(0) = x \end{aligned} \tag{3.50}$$

Here we need our control $u(t)$ to be adapted and càdlàg, and that we have a strong solution, $X^u(t)$, to (3.50). Our problem is then to maximize an expected value, with respect to our control $u(t)$, of the form:

$$J(u) = E\left[\int_0^T f(t, X(t), u(t))dt + g(X(T))\right] \tag{3.51}$$

and to have a solution to this we need f to be continuous, $g \in C^1(\mathbb{R})$, and $T < \infty$ to be deterministic. We also need this constraint:

$$E\left[\int_0^T f^-(t, X(t), u(t))dt + g^-(X(T))\right] < \infty \tag{3.52}$$

to hold for for all controls u, here $a^- = max\{-a, 0\}$.

Now we shall show how to solve these kinds of problems with the maximum principle.

First we need to define a Hamiltonian function H, given by:

$$\begin{aligned} H(t, x, u, p, g, r) &= f(t, x, u) + b^T(t, x, u)p + tr(\sigma^T(t, x, u)q) \\ &\quad + \sum_{j=1}^{l} \sum_{i=1}^{n} \int_{\mathbb{R}} \gamma_{ij}(t, x, u, z_j) r_{ij}(t, z)\nu_j(dz_j) \end{aligned} \tag{3.53}$$

and we need H to be differentiable with respect to x. Given this Hamiltonian function H, we get an adjoint equation in the unknown processes $p(t) \in \mathbb{R}^n$, $q(t) \in \mathbb{R}^{n\times m}$, and $r(t, z) \in \mathbb{R}^{n\times l}$, and this adjoint equation is a backward stochastic differential equation, given by:

$$\begin{aligned} dp(t) &= -\nabla_x H(t, X(t), u(t), p(t), q(t), r(t, \cdot))dt \\ &\quad + q(t)dB(t) + \int_{\mathbb{R}^l} r(t^-, z)\tilde{N}(dz, dz) \\ p(T) &= \nabla g(X(T)) \end{aligned} \tag{3.54}$$

A solution to our control problem, $\hat{u}(t)$, is optimal if our Hamiltonian function has a supremum for this control

$$H(t, X^{\hat{u}}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot)) = \sup_{u\in U} H(t, X^{\hat{u}}(t), u, \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot)) \quad (3.55)$$

for all t. Here $(\hat{p}(t), \hat{q}(t), \hat{r}(t,z))$ is a solution to our adjoint equation (3.54), and$X^{\hat{u}}(t)$ is the solution of (3.50) with this control, and U is the set of viable controls.

We also need $g(x)$ to be a concave function of x, and that

$$\hat{H}(x) = \max_{u\in U} H(t, x, u, \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot)) \quad (3.56)$$

exists and is a concave function of x for all $t \in [0,T]$. If we also have some integrability conditions, then $\hat{u}(t)$ is an optimal control.

For a proof of this, see [1, Ch. 3.2] and references herein.

3.7 Option Pricing

In this section we will show how to find arbitrage-free prices for an option, when the underlying $S(t)$ is an Itô-Lévy process. What we want to show is that for the process $S(t)$, we will get an arbitrage-free price for the option $H(S(T))$ if we compute

$$E_Q[e^{-\int_0^T r(s)ds} H(S(T))] = x \quad (3.57)$$

where Q is an ELMM for $\bar{S}(t) = e^{-\int_0^t r(s)ds} S(t)$.

If this price should allow an arbitrage for our option, then it means we could sell the option for this price, invest the money in the bank account and the risky asset $S(t)$, and at time T end up with something that is worth the same, or more, than what the option is worth.

So if we start with the value x, and we use a self-financing strategy to invest in the bank account $S_0(t) = e^{\int_0^t r(s)ds}$ and the risky asset $S(t)$, then the portfolio-value at time t will be:

$$\begin{aligned} V^\pi(t) &= \pi_0(t)S_0(t) + \pi_1(t)S(t) \\ &= x + \int_0^t \pi_0(s)dS_0(s) + \int_0^t \pi_1(s^-)dS(s) \end{aligned} \quad (3.58)$$

where $\{\pi_i(t)\}_{i=1,2}$ is what we have in the bank account and the risky asset respectively. So if the price x would allow for an arbitrage for our option, then we could find an investment strategy π, such that

$$V^\pi(T) \geq H(S(T)); \; P - a.s. \tag{3.59}$$

and we have a positive probability for strict inequality.

We could also discount the value of our investment, and we say

$$\bar{V}^\pi(t) = S_0^{-1}(t)V^\pi(t); \; \bar{V}^\pi(0) = x \tag{3.60}$$

Now we would like to look at the dynamics for this, but then we need to specify $S(t)$, which we define by:

$$dS(t) = \alpha(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}} \gamma(t,z)\tilde{N}(dt,dz); \; 0 \leq t \leq T \tag{3.61}$$

and we want this to be well defined. If we use the product rule for Itô-Lévy processes on $\bar{S}(t)$, we get that:

$$\begin{aligned} d\bar{S}(t) &= -S_0^{-1}(t)S(t)r(t)dt + S_0^{-1}(t)dS(t) \\ &= S_0^{-1}(t)(dS(t) - r(t)S(t)dt) \end{aligned} \tag{3.62}$$

and if we do the same on $\bar{V}^\pi(t)$, we get:

$$\begin{aligned} d(\bar{V}^\pi(t)) &= V^\pi(t)S_0^{-1}(t)(-r(t))dt + S_0^{-1}(t)\pi_0(t)S_0(t)r(t)dt \\ &\quad + S_0^{-1}(t)\pi_1(t)dS(t) \\ &= -\pi_0(t)r(t)dt - S_0^{-1}(t)r(t)\pi_1(t)S(t)dt + \pi_0(t)r(t)dt \\ &\quad + S_0^{-1}(t)\pi_1(t)dS(t) \\ &= \pi_1(t)S_0^{-1}(t)(dS(t) - r(t)S(t)dt) \\ &= \pi_1(t)d\bar{S}(t) \end{aligned} \tag{3.63}$$

So our investment strategy is self-financing for our discounted price-process as well, and if we integrate on both sides, we get:

$$\bar{V}^\pi(t) = x + \int_0^t \pi_1(s)d\bar{S}(s) \tag{3.64}$$

Since $\bar{S}(t)$ is a local martingale under Q by assumption, we get that $\bar{V}(t)$ is a local martingale under Q, under some integrability conditions. Now we assume that $\bar{V}(t)$ is lower bounded, if it wasn't we could borrow indefinitely, and then we could earn money a.s. with doubling strategies. When we assume that $\bar{V}(t) \geq -M$, for some constant M, we get that $\bar{V}(t)$ is a lower bounded local martingale, which is the same as a Q-supermartingale, and then

$$E_Q[\bar{V}(t)] \leq \bar{V}(0) = x \tag{3.65}$$

Now we can start showing that x is not an arbitrage price for our option, because if it was, then $V^\pi(T) \geq H(S(T))$ a.s., which implies:

$$E_Q[S_0^{-1}(t)(V^\pi(T) - H(S(T)))] \geq 0 \tag{3.66}$$

but from the the definition of x, and that $\bar{V}^\pi(t)$ is a supermartingale, we get that

$$E_Q[S_0^{-1}(t)(V^\pi(T) - H(S(T)))] \leq x - x = 0 \tag{3.67}$$

So our assumption is wrong, and we can't start with x and invest in such a way that we get more than the option value a.s., which means the price x will not allow for an arbitrage for our option. Similar arguments hold for shorting the option.

3.8 Fourier Transforms

In the previous section we saw that we got an arbitrage-free price by computing

$$E_Q[e^{-\int_0^T r(s)ds} H(S(T))] = x \tag{3.68}$$

and in this section we shall look at how to compute this value. Now we shall denote the price of the option by $P_Q(X, H)$, where Q is our ELMM, X is the underlying, and H is a function that specifies which option we look at, and we assume the risk free interest rate is zero, so we get:

$$P_Q(X, H) = E_Q[H(X)] \tag{3.69}$$

To compute this we will use Fourier transforms.

Definition 3 (Fourier Transform)**.** If we have a function $f \in L^1(\mathbb{R})$, then the Fourier transform of this function will be:

$$\hat{f}(y) = \int_{\mathbb{R}} f(x) e^{-xy} dx \tag{3.70}$$

and if $\hat{f} \in L^1(\mathbb{R})$ we can get back f again by the inverse Fourier transform, like this:

$$f(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{f}(y) e^{ixy} dy \tag{3.71}$$

So if H and $\hat{H}$ are in $L^1(\mathbb{R})$, then we can apply this to our option pricing problem like this:

$$\begin{aligned} E[H(X)] &= \frac{1}{2\pi} E\left[\int_{\mathbb{R}} \hat{H}(y) e^{iyX(T)} dy\right] \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{H}(y) E\left[e^{iyX(T)}\right] dy \end{aligned} \tag{3.72}$$

We also need some conditions to take the expectation inside the integral, like we need $E[\exp(iyX(T))]$ to exist, and the conditions of Fubini's Theorem must hold. The next thing we will notice is that $E[\exp(iyX)]$ is the characteristic function of X, so we need to be able to compute this.

For more details on this in the Lévy-process case, see [7].

3.9 Finding Investment Strategies by the Duality Method

An important problem in mathematical finance is finding optimal investment strategies, and in this section we will show how to find investment strategies that maximize the expected utility from an investment, and we shall do so with a duality method. So for an utility function $U(x)$, and starting value x, we would like to find an investment strategy π, that maximizes

$$E[U(X_x^\pi(T))] \tag{3.73}$$

where

$$X_x^\pi = x + \int_0^t \pi(s) dS(s) \tag{3.74}$$

is the value of our investment at time $0 \leq t \leq T$, $\pi(s)$ is the amount of risky assets we hold, and $S(s)$ is the value of our risky asset. The assumption here is that the risk free interest rate is zero, so this is the same as working with discounted values.

The duality approach is that instead of finding π^*, such that

$$u(x) = \sup_{\pi \in \mathcal{A}} E[U(X_x^\pi(T))] = E[U(X_x^{\pi^*}(T))] \tag{3.75}$$

we solve the dual problem. In the dual problem we introduce the convex dual of U, defined by:

$$V(y) := \sup_{x>0}\{U(x) - xy\};\ y > 0 \tag{3.76}$$

and we get back U again, by computing

$$U(x) = \inf_{y>0}\{V(y) + xy\};\ x > 0 \tag{3.77}$$

and we get this relationship between their derivatives:

$$U'(x) = y \Leftrightarrow x = -V'(y) \tag{3.78}$$

The dual problem is then to find $Q^* \in \mathcal{M}$(where $\mathcal{M}$ is the set of ELMM's), such that

$$v(y) = \inf_{Q\in\mathcal{M}} E\left[V\left(y\frac{dQ}{dP}\right)\right] = E\left[V\left(y\frac{dQ^*}{dP}\right)\right] \tag{3.79}$$

and we get under some conditions that π^* and Q^* exists, and are related by

$$U'(X_x^{\pi^*}(T)) = y\frac{dQ^*}{dP};\ y = u'(x) \tag{3.80}$$

and

$$X_x^{\pi^*}(T) = -V'\left(y\frac{dQ^*}{dP}\right);\ x = -v'(y) \tag{3.81}$$

What this means is that instead of finding an optimal investment strategy π^*, we can find an optimal ELMM Q^*, which will give us our optimal final wealth $X_x^{\pi^*}(T)$, and then we need to find an investment strategy π^* that generates this wealth.

For proofs and more theory about the duality method see [2] and references herein. In this paper they also include the Inada conditions:

$$\begin{aligned} U'(0) &= \lim_{x\to 0^+} U'(x) = \infty \\ U'(\infty) &= \lim_{x\to\infty} U'(x) = 0 \end{aligned}$$

but the utility function we are interested in will not satisfy these, and we will show they are not required for our problem by solving the maximization problem directly as well. With the Inada conditions we will get this relationship between U and the convex dual V:

$$V(0) = -\infty,\ V'(\infty) = 0 \text{ and } V(\infty) = U(0) \tag{3.82}$$

4 HJM Model Driven by an Itô-Lévy Process

In this chapter we will expand the classical HJM model, to a model where the noise comes from both a Brownian motion and a compensated Poisson random measure. First we will describe our model, then we will find an equation that describes the set of ELMM's for our HJM model, and at last we will construct a LIBOR model from this HJM model. Everything we do in this chapter will be similar to what is done in Chapter 2, when our noise was driven by only a Brownian motion.

4.1 Noise from Poisson Process

As in Chapter 2, we will describe forward rates $f(t,s)$ at time s as seen from an earlier time point t. The difference is that we now add some terms which corresponds to the noise coming from the compensated Poisson process, so the forward rates $f(t,s)$ are given by:

$$\begin{aligned} f(t,s) &= f(0,s) + \int_0^t \alpha(v,s)dv + \int_0^t \sigma(v,s)dB(v) + \\ & \int_0^t \int_{\mathbb{R}} \gamma(s,v,z)\tilde{N}(dv,dz) \end{aligned} \tag{4.1}$$

For this to be well defined we need our processes $\alpha(v,s)$, $\sigma(v,s)$ and $\gamma(v,s,z)$ to be $\mathcal{F}_v$-predictable for $s \geq t$. We also want this integrability condition to hold:

$$E\left[\int_0^t \left\{\alpha^2(v,s) + \sigma^2(v,s) + \int_{\mathbb{R}} \gamma^2(v,s,z)\nu(dz)\right\} dv\right] < \infty \tag{4.2}$$

Then the integrals $\int_0^t \sigma(v,s)dB(v)$ and $\int_0^t \int_{\mathbb{R}} \gamma(v,s,z)\tilde{N}(dv,dz)$ are martingales under the probability measure P.

As in Chapter 2, our zero coupon price is denoted by $P(t,T)$, and it is given by:

$$
\begin{aligned}
P(t,T) &= \exp\left(-\int_t^T f(t,s)ds\right) \\
&= \exp\left(-\int_t^T f(0,s)ds + \int_t^T \int_0^t \alpha(v,s)dvds\right. \\
&\quad \left. + \int_t^T \int_0^t \sigma(v,s)dB(v)ds + \int_t^T \int_0^t \int_{\mathbb{R}} \gamma(v,s,z)\tilde{N}(dv,dz)ds\right)
\end{aligned}
\tag{4.3}
$$

Our next step will be as in Chapter 2, where we found an expression for $P(t,T)$ which we could use Itô's formula directly on. First we add and subtract $\int_t^T f(s,s)ds$ in the exponent of (4.3), and then we use a stochastic Fubini's theorem to interchange the limits, like this:

$$
\begin{aligned}
P(t,T) &= \exp\left(-\int_t^T f(s,s)ds + \int_t^T (f(s,s) - f(t,s))ds\right) \\
&= \exp\left(-\int_t^T f(s,s)ds + \int_t^T \left\{\int_t^s \alpha(v,s)dv + \int_t^s \sigma(v,s)dB(v)\right.\right. \\
&\quad \left.\left. + \int_t^s \int_{\mathbb{R}} \gamma(v,s,z)\tilde{N}(dv,dz)\right\} ds\right) \\
&= \exp\left(-\int_t^T f(s,s)ds + \int_t^T \int_v^T \alpha(v,s)dsdv\right. \\
&\quad \left. + \int_t^T \int_v^T \sigma(v,s)dsdB(v) + \int_t^T \int_{\mathbb{R}} \int_v^T \gamma(v,s,z)ds\tilde{N}(dv,dz)\right)
\end{aligned}
\tag{4.4}
$$

Now we simplify this by defining $\bar{\sigma}(t,T) = \int_t^T \sigma(t,s)ds$, $\bar{\alpha}(t,T) = \int_t^T \alpha(t,s)ds$ and $\bar{\gamma}(t,T,z) = \int_t^T \gamma(t,s,z)ds$, and we note that these processes are $\mathcal{F}_t$-predictable. Using this we can simplify (4.4) to:

$$
\begin{aligned}
P(t,T) &= \exp\left(-\int_t^T f(s,s)ds + \int_t^T \bar{\alpha}(v,T)dv + \int_t^T \bar{\sigma}(v,T)dB(v)\right. \\
&\quad \left. + \int_t^T \int_{\mathbb{R}} \bar{\gamma}(v,T,z)\tilde{N}(dv,dz)\right)
\end{aligned}
\tag{4.5}
$$

As in the Brownian motion case, we are interested in discounted bond prices:

$$\begin{aligned}\tilde{P}(t,T) &= \frac{P(t,T)}{\beta(t)} \\ &= \exp\left(-\int_0^T f(s,s)ds + \int_t^T \bar{\alpha}(v,T)dv + \int_t^T \bar{\sigma}(v,T)dB(v)\right. \\ &\quad \left. + \int_t^T \int_{\mathbb{R}} \bar{\gamma}(v,T,z)\tilde{N}(dv,dz)\right) \qquad (4.6)\end{aligned}$$

The next step is then to find the dynamics, $d\tilde{P}(t,T)$, and a new probability measure Q, which $\tilde{P}(t,T)$ is a local martingale under.

If we say $\tilde{P}(t,T) = f(X(t))$ where $f(x) = \exp(x)$, and

$$X(t) = -\int_0^T f(s,s)ds + \int_t^T \bar{\alpha}(v,T)dv + \int_t^T \bar{\sigma}(v,T)dB(v) + \int_t^T \int_{\mathbb{R}} \bar{\gamma}(v,T,z)\tilde{N}(dv,dz) \qquad (4.7)$$

then we get this expression for $dP(t,T)$ by Itô's formula:

$$\begin{aligned}dP(t,T) &= P(t,T)\left(-\bar{\alpha}(t,T) + \frac{1}{2}\bar{\sigma}^2(t,T)\right)dt - P(t,T)\bar{\sigma}(t,T)dB(t) \\ &\quad + \int_{\mathbb{R}} P(t,T)\left[\exp(-\bar{\gamma}(t,T,z)) - 1 + \bar{\gamma}(t,T,z)\right]\nu(dz)dt \\ &\quad + \int_{\mathbb{R}} P(t^-,T)\left[\exp(-\bar{\gamma}(t,T,z)) - 1\right]\tilde{N}(dt,dz) \qquad (4.8)\end{aligned}$$

Now we have found the dynamics of our discounted bond price process, and then it is easy to separate the drift term, which will be:

$$P(t,T)\left\{-\bar{\alpha}(t,T) + \frac{1}{2}\bar{\sigma}^2(t,T) + \int_{\mathbb{R}}\left[\exp(-\bar{\gamma}(t,T,z)) - 1 + \bar{\gamma}(t,T,z)\right]\nu(dz)\right\} \qquad (4.9)$$

Then from Girsanov's theorem, we can define a new probability measure by:

$$Q(A) = E[\mathbb{1}(A)Z(T)] \qquad (4.10)$$

Where $Z(T)$ is given by:

$$
\begin{aligned}
Z(t) \quad := \quad & \exp\Bigg(-\int_0^t q(s)dB(s) - \frac{1}{2}\int_0^t q^2(s)ds \\
& + \int_0^t \int_{\mathbb{R}} \ln(1-\theta(s,z))\tilde{N}(ds,dz) \\
& + \int_0^t \int_{\mathbb{R}} [\ln(1-\theta(s,z)) + \theta(s,z)]\nu(dz)ds\Bigg)
\end{aligned}
$$

$$E[Z(T)] = 1$$

Then we know from Girsanov's theorem that $\theta(t,z)$ and $q(t)$ need to satisfy this equation

$$
\begin{aligned}
& -\bar{\alpha}(t,T) + \frac{1}{2}\bar{\sigma}^2(t,T) + \int_{\mathbb{R}} \left[\exp(-\bar{\gamma}(t,T,z)) - 1 + \bar{\gamma}(t,T,z)\right]\nu(dz) \\
= \quad & -\bar{\sigma}(t,T)q(t) + \int_{\mathbb{R}} \theta(t,z)[\exp(-\bar{\gamma}(t,T,z)) - 1]\nu(dz) \qquad (4.11)
\end{aligned}
$$

for Q to be an ELMM.

Now we have described our set of ELMM's, the problem then will be to find the right kind of probability measure for a given situation. We also need our functions $q(t)$ and $\theta(t,z)$ to be predictable, but as before, this is already taken care of, since we have assumed our functions $\bar{\alpha}(t,T)$, $\bar{\sigma}(t,T)$ and $\bar{\gamma}(t,T,z)$ to be $\mathcal{F}_t$-predictable for $t \leq T$.

If we choose processes $q(t)$ and $\theta(t,z)$ that solves (4.11), then (4.8) can be written like this:

$$
\begin{aligned}
dP(t,T) \quad = \quad & -P(t,T)\bar{\sigma}(t,T)dB_Q(t) \\
& + \int_{\mathbb{R}} P(t^-,T)\left[\exp(-\bar{\gamma}(t,T,z)) - 1\right]\tilde{N}_Q(dt,dz) \qquad (4.12)
\end{aligned}
$$

Here

$$dB_Q(t) = dB(t) + q(t)dt \qquad (4.13)$$

is a Brownian motion with respect to Q, and

$$\tilde{N}_Q(dt,dz) = \tilde{N}(dt,dz) + \theta(t,z)\nu(dz)dt \qquad (4.14)$$

is the Q-compensated Poisson random measure of $N(\cdot,\cdot)$, in the sense that

$$M(t) = \int_0^t \int_{\mathbb{R}} \phi(s,z)\tilde{N}_Q(ds,dz);\ 0 \leq t \leq T \tag{4.15}$$

is a local $Q-martingale$ for all predictable processes $\phi(s,z)$ such that

$$\int_0^T \int_{\mathbb{R}} \phi^2(t,z)\theta^2(t,z)\nu(dz)dt < \infty\ a.s. \tag{4.16}$$

4.2 LIBOR Model with Jumps

Now we shall do the same as in Chapter 2, where we defined the LIBOR model by:

$$L(t,T) = \frac{P(t,T)}{\delta P(t,T+\delta)} - \frac{1}{\delta} \tag{4.17}$$

To find a suitable expression for this, we first need to rewrite our expression for $P(t,T)$. The first thing we do is to use (4.5), with $t=0$, to get our initial forward curve

$$\begin{aligned} P(0,T) &= \exp\Big(-\int_0^T f(s,s)ds + \int_0^T \bar{\alpha}(v,T)dv + \int_0^T \bar{\sigma}(v,T)dB(v) \\ &\quad + \int_0^T \int_{\mathbb{R}} \bar{\gamma}(v,T,z)\tilde{N}(dv,dz)\Big) \end{aligned} \tag{4.18}$$

then using (4.5) and (4.18) we get this expression for $P(t,T)$:

$$\begin{aligned} P(t,T) &= P(0,T)\exp\Big(\int_0^t f(s,s)ds - \int_0^t \bar{\alpha}(v,T)dv - \int_0^t \bar{\sigma}(v,T)dB(v) \\ &\quad - \int_0^t \int_{\mathbb{R}} \bar{\gamma}(v,T,z)\tilde{N}(dv,dz)\Big) \end{aligned} \tag{4.19}$$

Then we insert (4.19) into (4.17) and we get this expression for $L(t,T)$:

$$\begin{aligned} L(t,T) &= \frac{P(0,T)}{\delta P(0,T+\delta)}\exp\Big(\int_0^t \bar{\alpha}(v,T+\delta) - \bar{\alpha}(v,T)dv + \int_0^t \bar{\sigma}(v,T+\delta) - \bar{\sigma}(v,T)dB(v) \\ &\quad + \int_0^t \int_{\mathbb{R}} \bar{\gamma}(v,T+\delta,z) - \bar{\gamma}(v,T,z)\tilde{N}(dv,dz)\Big) - \frac{1}{\delta} \end{aligned} \tag{4.20}$$

and this will be our expression for our forward rates in the LIBOR model.

5 ELMM's, Option Prices and Investment Strategies

In this chapter we will show how to calculate option prices for our LIBOR model. There are two problems with calculating option prices when we work with Itô-Lévy processes, the first is that we will in general have incomplete markets in this setting, so we will get several arbitrage-free prices, one for each ELMM, and then we need to choose which one to use. The second problem is that when we model with jumps, we will get a much more complex model and we will typically not find analytical solutions to our problems. Because of this we will need to use numerical methods to compute our option prices. In this chapter we will first find different ELMM's and then look at how to compute option prices corresponding to these measures. At last we will use the duality method to find investment strategies for the HJM-model.

The main results in this chapter is the measures for minimal quadratic distance, the minimal entropy measure, the two Esscher transforms, and showing the link between the minimal entropy measure and the simple return Esscher transformed martingale measure. We will also show how to compute the price a European call option, and we will use the duality method to find optimal investment strategies when have an exponential utility function.

5.1 Option Prices

In this section we will specify how the option price will be defined in our specific case, and go through how our approach to computing this option price will be.

In our case, the underlying is $L(t,T)$, and we have shown we can compute an arbitrage free option-price by computing the expectation under an ELMM, like this:

$$\begin{aligned} P_Q(L,H) &= E_Q\left[\exp\left(-\int_0^{T+\delta} r(s)ds\right) H(L(T,T))\right] \\ &= E\left[Z(T)\exp\left(-\int_0^{T+\delta} r(s)ds\right) H(L(T,T))\right] \end{aligned} \tag{5.1}$$

As before $H(x)$ specifies which kind of option we are interested in, and in our case we will have a European call option at time $T+\delta$, so we get the difference between $L(T,T)$ and K, when $L(T,T) > K$. In that case H is given by $H(x) = \max\{0, x-k\}$. We also recall that $Z(T)$ is given by:

$$\begin{aligned} Z(t) &= \exp\Big(-\int_0^t q(s)dB(s) - \frac{1}{2}\int_0^t q^2(s)ds + \int_0^t \int_{\mathbb{R}} \ln(1-\theta(s,z))\tilde{N}(ds,dz) \\ &\quad + \int_0^t \int_{\mathbb{R}} [\ln(1-\theta(s,z)) + \theta(s,z)]\nu(dz)ds \Big) \end{aligned} \tag{5.2}$$

where $q(t)$ and $\theta(t,z)$ must satisfy equation (4.11) for Q to be an ELMM. When we choose to use this kind of measure change, and not a change of measure such that $L(t,T)$ is a martingale, we do it because that is what is done in the case when we only have a Brownian motion. The results we gain about our measure change are obtained using quite general processes, so it should work for a different measure as well.

In this chapter we shall first find probability measures Q, which will be done by two different methods. First we shall find the ELMM which differs the least from our original measure P, with respect to some distance measure. The next method we will use is the Esscher transform, which we have described earlier. The reason for why we are interested in finding several ELMM's is that we might find useful similarities between the measures, which can only be found by calculating several different measures, and we might link the measures to other problems, as we have seen in the duality method described earlier. After we have found these measures, we shall use the Fourier transform to compute the price of a European call option when we use a general measure change.

5.2 Minimal Distance Measures

In the next two subsections we will find probability measures by minimizing the distance between our ELMM and our "real world" probability measure P with respect to some function f. The reason for why we want to look at measures like this, is that we can look at this as measures that closely resembles our original measure in terms of probability. We also saw in our description of our duality method that this can be linked to the utility maximization problem. We will find two such measures, one for the function $f(x) = x^2$, and one for the function $g(x) = x\ln(x)$.

5.2.1 Minimal Quadratic Distance

In this section we will look at what we call the minimal quadratic distance measure, and for a general function our problem is to find a probability measure Q that solves:

$$\inf_{Q \in \mathcal{M}} E_P \left[f \left(\frac{dQ}{dP} \right) \right] \tag{5.3}$$

where $\mathcal{M}$ is the set of all ELMM's and $f \in C^1$ is a convex function. In this section we want to find the minimal quadratic distance measure, and then we look at the function $f(x) = x^2$.

The process subject to minimization is $\frac{d(Q|\mathcal{F}_t)}{d(P|\mathcal{F}_t)} = G(t)$, where $G(t)$ is given by:

$$dG(t) = -G(t)u(t)dB(t) - G(t^-) \int_{\mathbb{R}} \theta(t,z)\tilde{N}(dt,dz); \; G(0) = 1 \tag{5.4}$$

and for Q to be a ELMM, we need that $u(t)$ and $\theta(t,z)$ satisfy equation (4.11):

$$\begin{aligned} &-\bar{\alpha}(t,T) + \frac{1}{2}\bar{\sigma}^2(t,T) + \int_{\mathbb{R}} [\exp(-\bar{\gamma}(t,T,z)) - 1 + \bar{\gamma}(t,T,z)]\nu(dz) \\ &= -\bar{\sigma}(t,T)u(t) + \int_{\mathbb{R}} \theta(t,z)[\exp(-\bar{\gamma}(t,T,z)) - 1]\nu(dz) \end{aligned}$$

We see this is a control problem, in the variables $u(t)$ and $\theta(t,z)$, and to solve this problem we will use the maximum principle. The first thing we need to do is to find the Hamiltonian function H, which in our case will be:

$$H(t,g,u,\theta,p,q,r) = -guq - g \int_{\mathbb{R}} \theta(t,z)r(t,z)\nu(dz) \tag{5.5}$$

and our adjoint equation will be:

$$\begin{aligned} dp(t) &= \left[u(t)q(t) + \int_{\mathbb{R}} \theta(t,z)r(t,z)\nu(dz) \right] dt \\ &\quad + q(t)dB(t) + \int_{\mathbb{R}} r(t^-,z)\tilde{N}(dt,dz) \qquad (5.6) \\ p(T) &= 2G(T) \end{aligned}$$

To solve this problem, we will first reduce our number of variables by solving equation (4.11) for $u(t)$, and we get that:

$$\begin{aligned} u(t) &= [\bar{\alpha}(t,T) - \frac{1}{2}\bar{\sigma}^2(t,T) + \int_{\mathbb{R}} [(\exp(-\bar{\gamma}(t,T,z)) - 1)(\theta(t,z) - 1) \\ &\quad -\bar{\gamma}(t,T,z)]\nu(dz)]/\bar{\sigma}(t,T) \end{aligned} \tag{5.7}$$

and for this solution to be valid, we need that $\bar{\sigma}(t,T) \neq 0$. If we then insert the solution of (5.7) into (5.5) we get:

$$H(t,G(t),u(t),\theta(t,z),p(t),q(t),r(t,z)) = -G(T)\Bigg[(\bar{\alpha}(t,T) - \frac{1}{2}\bar{\sigma}^2(t,T))\cdot$$
$$\frac{q(t)}{\bar{\sigma}(t,T)} + \int_{\mathbb{R}}\{[(\exp(-\bar{\gamma}(t,T,z)) - 1)\frac{q(t)}{\bar{\sigma}(t,T)} + r(t,z)]\theta(t,z)$$
$$+[1-\bar{\gamma}(t,T,z) - \exp(-\bar{\gamma}(t,T,z))]\frac{q(t)}{\bar{\sigma}(t,T)}\}\nu(dz)\Bigg] \tag{5.8}$$

The next thing we do is to minimize this with respect to $\theta(t,z)$, and if there exists a function $\hat{\theta}(t,z)$ that minimizes this, we need that $(\nabla_\theta H)_{\theta=\hat{\theta}} = 0$, and this is the same as:

$$(\exp(-\bar{\gamma}(t,T+\delta,z)) - 1)\frac{q(t)}{\bar{\sigma}(t,T+\delta)} + r(t,z)) = 0;\ 0 \leq t \leq T \tag{5.9}$$

and this should hold for optimal $(\hat{p},\hat{q},\hat{r})$. If we insert (5.7) and (5.9) into (5.6), we will get this backward stochastic differential equation(BSDE):

$$\begin{aligned} d\hat{p}(t) &= \frac{\hat{q}(t)}{\bar{\sigma}(t,T)}\Bigg[\bar{\alpha}(t,T) - \frac{1}{2}\bar{\sigma}^2(t,T) + \int_{\mathbb{R}}(1-\bar{\gamma}(t,T,z) \\ &\quad - \exp(-\bar{\gamma}(t,T,z)))\nu(dz)\Bigg]\,dt \\ &\quad + \hat{q}(t)dB(t) + \int_{\mathbb{R}}(1-\exp(-\bar{\gamma}(t,T,z)))\frac{\hat{q}(t^-)}{\bar{\sigma}(t,T)}\tilde{N}(dt,dz) \qquad (5.10) \\ \hat{p}(T) &= 2G(T) \end{aligned}$$

To find a solution to this, we try with $p(t)$ on the form $p(t) = \phi_t G(t)$, and we assume ϕ_t to be of the form $d\phi_t = \phi_t' dt$, so we have no integrals with respect to the Compensated Poisson random measure or the Brownian motion, and we also need ϕ_t to be $\mathcal{F}_t$-predictable. If we use Itô's formula on $p(t)$, we get:

$$\begin{aligned} dp(t) &= -\frac{\phi_t G(t)}{\bar{\sigma}(t,T)}\Bigg[\bar{\alpha}(t,T) - \frac{1}{2}\bar{\sigma}^2(t,T) + \int_{\mathbb{R}}\{(\bar{\gamma}(t,T,z)-1) \\ &\quad (\theta(t,z)-1) - \bar{\gamma}(t,T,z)\}\nu(dz)\Bigg]dB(t) - \phi_t G(t^-)\int_{\mathbb{R}}\theta(t,z)\tilde{N}(dt,dz) \\ &\quad + G(t)\phi_t' dt \end{aligned} \tag{5.11}$$

Then we can compare (5.10) with (5.11), and we get these three equations:

$$
\begin{aligned}
q(t) &= -\frac{\phi_t G(t)}{\bar{\sigma}(t,T)} \Big[\bar{\alpha}(t,T) - \frac{1}{2}\bar{\sigma}^2(t,T) + \int_{\mathbb{R}} \{(\bar{\gamma}(t,T,z) - 1) \\
& \quad (\theta(t,z) - 1) - \bar{\gamma}(t,T,z)\}\nu(dz) \Big] \\
&= -\phi_t G(t) u(t)
\end{aligned}
\tag{5.12}
$$

$$
\frac{q(t^-)}{\bar{\sigma}(t,T)} \int_{\mathbb{R}} \{1 - \exp(-\bar{\gamma}(t,T,z))\} \tilde{N}(t,dz) = -\phi_t G(t^-) \int_{\mathbb{R}} \theta(t,z) \tilde{N}(t,dz)
\tag{5.13}
$$

and

$$
\begin{aligned}
G(t)\phi_t' &= \frac{q(t)}{\sigma(t,T)} \Big[\bar{\alpha}(t,T) - \frac{1}{2}\bar{\sigma}^2(t,T) \\
& \quad + \int_{\mathbb{R}} (1 - \exp(-\bar{\gamma}(t,T,z)) - \bar{\gamma}(t,T,z))\nu(dz) \Big]
\end{aligned}
\tag{5.14}
$$

To study equation (5.13), we will use the Itô-Lévy Isometry to show that if two processes of the form of (5.13) are equal, then their integrands must be equal[4]. We recall from the Itô-Lévy Isometry that if $X(t)$ is of the form:

$$
X(t) = \int_0^t \int_{\mathbb{R}} \theta(s,z) \tilde{N}(ds,dz)
\tag{5.15}
$$

and $E\left[\int_0^t \int_{\mathbb{R}} \theta^2(s,z)\nu(dz)ds\right] < \infty$, then we have this equality:

$$
E[X^2(t)] = E\left[\int_0^t \int_{\mathbb{R}} \theta^2(s,z)\nu(dz)ds\right]
\tag{5.16}
$$

If we define X(t), $0 \le t \le T$, by two different processes, like this:

$$
X(t) = \int_0^t \int_{\mathbb{R}} \theta(s,z) \tilde{N}(ds,dz)
\tag{5.17}
$$

and

$$
X(t) = \int_0^t \int_{\mathbb{R}} \gamma(s,z) \tilde{N}(ds,dz)
\tag{5.18}
$$

[4] We have done the same for the integral with respect to the Brownian motion, but we assume the result is known in that case at least, the proof is similar if not.

and we define $Y(t)$ as:

$$Y(t) = \int_0^t \int_{\mathbb{R}} (\theta(s,z) - \gamma(s,z))\tilde{N}(ds,dz) = X(t) - X(t) = 0 \tag{5.19}$$

then we get from the Itô-Lévy Isometry that:

$$E[Y^2(t)] = E\left[\int_0^t \int_{\mathbb{R}} (\theta(s,z) - \gamma(s,z))^2 \nu(dz)ds\right] = 0 \tag{5.20}$$

So we must have $\theta(t,z) = \gamma(t,z)$ for $0 \leq t \leq T$ a.s. If we use this result we can rewrite equation (5.13) on the form:

$$\frac{q(t^-)}{\bar{\sigma}(t,T)}(1 - \exp(-\bar{\gamma}(t,T,z))) = -\phi_t G(t^-)\theta(t,z) \tag{5.21}$$

or:

$$q(t^-) = -\frac{\phi_t G(t^-)\bar{\sigma}(t,T)\theta(t,z)}{1 - \exp(-\bar{\gamma}(t,T,z))} \tag{5.22}$$

Now we compare equation (5.12) with equation (5.22), which will give us an equation for $\theta(t,z)$:

$$\begin{aligned} \frac{\phi_t G(t)\bar{\sigma}(t,T)\theta(t,z)}{1 - \exp(-\bar{\gamma}(t,T,z))} &= \frac{\phi_t G(t)}{\bar{\sigma}(t,T)}\Bigg[\bar{\alpha}(t,T) - \frac{1}{2}\bar{\sigma}^2(t,T) + \int_{\mathbb{R}}\{(\bar{\gamma}(t,T,z) - 1) \\ &\qquad (\theta(t,z) - 1) - \bar{\gamma}(t,T,z)\}\nu(dz)\Bigg] \end{aligned} \tag{5.23}$$

and this can be simplified to:

$$\begin{aligned} \frac{\bar{\sigma}(t,T)\theta(t,z)}{1 - \exp(-\bar{\gamma}(t,T,z))} &= \frac{1}{\bar{\sigma}(t,T)}\Bigg[\bar{\alpha}(t,T) - \frac{1}{2}\bar{\sigma}^2(t,T) + \int_{\mathbb{R}}\{(\bar{\gamma}(t,T,z) - 1) \\ &\qquad (\theta(t,z) - 1) - \bar{\gamma}(t,T,z)\}\nu(dz)\Bigg] \end{aligned} \tag{5.24}$$

We can't find a solution to this analytically, but if it has a solution, we can call the solution to this for $\hat{\theta}$, and we can insert this into (5.22) and we get an expression for $q(t)$ where ϕ_t is the only unknown. If we insert the expression for $q(t)$ into (5.14) we will get this differential equation for ϕ_t

$$\begin{aligned} \phi_t' &= -\frac{\phi_t\hat{\theta}(t,z)}{1 - \exp(-\bar{\gamma}(t,T,z))}\Bigg[\bar{\alpha}(t,T) - \frac{1}{2}\bar{\sigma}^2(t,T) \\ &\qquad + \int_{\mathbb{R}}(1 - \exp(-\bar{\gamma}(t,T,z)) - \bar{\gamma}(t,T,z))\nu(dz)\Bigg] \end{aligned} \tag{5.25}$$

And since $p(T) = 2G(T)$ we get that $\phi_T = 2$, and our solution to (5.25) will be:

$$\begin{aligned}\phi_t &= 2\exp\left(\int_t^T \frac{\hat{\theta}(s,z)}{1-\exp(-\bar{\gamma}(s,T,z))}\left[\bar{\alpha}(s,T) - \frac{1}{2}\bar{\sigma}^2(s,T)\right.\right.\\ &\quad \left.\left. + \int_{\mathbb{R}}(1-\exp(-\bar{\gamma}(s,T,z)) - \bar{\gamma}(s,T,z))\nu(dz)\right] ds\right) \qquad (5.26)\end{aligned}$$

The problem now is that ϕ_t is defined by an integral from t to T, so ϕ_t will not be $\mathcal{F}_t$-adapted, therefore we need $\hat{\theta}$, $\bar{\gamma}$, $\bar{\alpha}$ and $\bar{\sigma}$ to be deterministic.

Our conclusion is that the measure that solves

$$\inf_{Q\in\mathcal{M}} E_P\left[G^2(T)\right]$$

where

$$\begin{aligned}G(t) &:= \exp\left(-\int_0^t u(s)dB(s) - \frac{1}{2}\int_0^t u^2(s)ds\right.\\ &\quad + \int_0^t\int_{\mathbb{R}} \ln(1-\theta(s,z))\tilde{N}(ds,dz)\\ &\quad \left. + \int_0^t\int_{\mathbb{R}}\{\ln(1-\theta(s,z)) + \theta(s,z)\}\nu(dz)ds\right)\end{aligned}$$

is given by the equations:

$$\begin{aligned}\frac{\bar{\sigma}(t,T)\theta(t,z)}{1-\exp(-\bar{\gamma}(t,T,z))} &= \frac{1}{\bar{\sigma}(t,T)}\left[\bar{\alpha}(t,T) - \frac{1}{2}\bar{\sigma}^2(t,T) + \int_{\mathbb{R}}\{(\bar{\gamma}(t,T,z)-1)\right.\\ &\quad \left.(\theta(t,z)-1) - \bar{\gamma}(t,T,z)\}\nu(dz)\right]\end{aligned}$$

and

$$\begin{aligned}u(t) &= [\bar{\alpha}(t,T) - \frac{1}{2}\bar{\sigma}^2(t,T) + \int_{\mathbb{R}}[(\exp(-\bar{\gamma}(t,T,z)) - 1)(\theta(t,z)-1)\\ &\quad -\bar{\gamma}(t,T,z)]\nu(dz)]/\bar{\sigma}(t,T)\end{aligned}$$

when all our integrands are deterministic.

5.2.2 Minimal Entropy Martingale Measure

Now we are interested in the same as in the previous section, except that we want to look at the function $f(x) = x\ln(x)$ instead of $g(x) = x^2$, the measure we find with this function is called the minimal entropy martingale measure(MEMM).

As before we will state what our Hamiltonian function and adjoint equation will be, but we see that the only place where our function g comes into play is for the boundary condition in the adjoint equation, so the setup for our problem will be much the same. Therefore we shall just state what it will look like:

$$\begin{aligned} dp(t) &= \left[u(t)q(t) + \int_{\mathbb{R}} \theta(t,z)r(t,z)\nu(dz)\right] dt \\ &\quad + q(t)dB(t) + \int_{\mathbb{R}} r(t^-,z)\tilde{N}(dt,dz) \\ p(T) &= \ln(G(T)) + 1 \end{aligned} \tag{5.27}$$

$$\begin{aligned} u(t) &= [\bar{\alpha}(t,T) - \frac{1}{2}\bar{\sigma}^2(t,T) + \int_{\mathbb{R}} [(\exp(-\bar{\gamma}(t,T,z)) - 1)(\theta(t,z) - 1) \\ &\quad - \bar{\gamma}(t,T,z)]\nu(dz)]/\bar{\sigma}(t,T) \end{aligned}$$

$$\begin{aligned} &H(t,G(t),u(t),\theta(t,z),p(t),q(t),r(t,z)) = -G(T)\Bigg[(\bar{\alpha}(t,T) - \frac{1}{2}\bar{\sigma}^2(t,T))\cdot \\ &\frac{q(t)}{\bar{\sigma}(t,T)} + \int_{\mathbb{R}} \{(\exp(-\bar{\gamma}(t,T,z)) - 1)(\frac{q(t)}{\bar{\sigma}(t,T)} + r(t,z))\theta(t,z) \\ &+ [1 - \bar{\gamma}(t,T,z) - \exp(-\bar{\gamma}(t,T,z))]\frac{q(t)}{\bar{\sigma}(t,T)}\}\nu(dz)\Bigg] \end{aligned}$$

$$(\exp(-\bar{\gamma}(t,T,z)) - 1)\frac{q(t)}{\bar{\sigma}(t,T)} + r(t,z)) = 0;\ 0 \le t \le T$$

So now the problem is to solve this set of equations, and as before we will guess a solution to $p(t)$, and see that this solves our problem. So the solution we guess at now is $p(t) = \ln(G(t)) + \phi_t$, and again we assume $d\phi_t = \phi'_t dt$, and that ϕ_t is $\mathcal{F}_t$-predictable. The first thing we use is that $\ln(G(t))$ is given by:

$$\begin{aligned}\ln(G(t)) &= -\int_0^t u(s)dB(s) - \frac{1}{2}\int_0^t u^2(s)ds + \int_0^t \int_{\mathbb{R}} \ln(1-\theta(s,z))\tilde{N}(ds,dz) \\ &\quad + \int_0^t \int_{\mathbb{R}} \{\ln(1-\theta(s,z)) + \theta(s,z)\}\nu(dz)ds \qquad (5.28)\end{aligned}$$

and given this, we have two different ways to express $dp(t)$ by, so we will look at these, and compare terms. We have

$$\begin{aligned}dp(t) &= \left[u(t)q(t) + \int_{\mathbb{R}} \theta(t,z)(1-\exp(-\bar{\gamma}(t,T,z)))\frac{q(t)}{\bar{\sigma}(t,T)}\nu(dz)\right]dt \\ &\quad + q(t)dB(t) + \int_{\mathbb{R}} (1-\exp(-\bar{\gamma}(t,T,z)))\frac{q(t^-)}{\bar{\sigma}(t,T)}\tilde{N}(dt,dz)\end{aligned} \qquad (5.29)$$

and

$$\begin{aligned}dp(t) &= -u(t)dB(t) - \frac{1}{2}u^2(t)dt + \int_{\mathbb{R}} \ln(1-\theta(t,z))\tilde{N}(dt,dz) \\ &\quad + \int_{\mathbb{R}} \{\ln(1-\theta(t,z)) + \theta(t,z)\}\nu(dz)dt + \phi'_t dt \qquad (5.30)\end{aligned}$$

if we compare these two expressions, we get these equations:

$$q(t) = -u(t) \qquad (5.31)$$

$$\ln(1-\theta(t,z)) = (1-\exp(-\bar{\gamma}(t,T,z)))\frac{q(t^-)}{\bar{\sigma}(t,T)} \qquad (5.32)$$

and

$$\begin{aligned}&u(t)q(t) + \int_{\mathbb{R}} \theta(t,z)(1-\exp(-\bar{\gamma}(t,T,z)))\frac{q(t)}{\bar{\sigma}(t,T)}\nu(dz) \\ &= -\frac{1}{2}u^2(t) + \phi'_t + \int_{\mathbb{R}} \{\ln(1-\theta(t,z)) + \theta(t,z)\}\nu(dz) \qquad (5.33)\end{aligned}$$

and we also have our earlier equation for $u(t)$, so this means we have four equations, and four unknowns, $q(t)$, $u(t)$, $\theta(t,z)$ and ϕ'_t, so it should be possible to find a solution to this set of equations if our equations are linearly independent.

To solve this set of equations, we first see that $u(t)$ and $\theta(t,z)$ are defined by (5.7) and (5.32). Then we insert (5.31) and (5.32) into (5.33), which will give us this equation for ϕ'_t:

$$\begin{aligned}\phi'_t &= -\frac{1}{2}u^2(t) + \int_{\mathbb{R}}(1-\theta(t,z))(1-\exp(-\bar{\gamma}(t,T+\delta,z)))\frac{u(t)}{\bar{\sigma}(t,T+\delta)} \\ &\quad -\theta(t,z)\nu(dz) \qquad (5.34)\end{aligned}$$

and we need that $\phi_T = 1$, so we will again need that u, θ, $\bar{\gamma}$, $\bar{\alpha}$ and $\bar{\sigma}$ are deterministic. The solutions $\theta(t,z)$ and $u(t)$ to these equations will give us the MEMM.

Our conclusion is that the measure that solves

$$\inf_{Q\in\mathcal{M}} E_P\left[G(T)\ln(G(T))\right]$$

where

$$\begin{aligned}G(t) &:= \exp\Bigg(-\int_0^t u(s)dB(s) - \frac{1}{2}\int_0^t u^2(s)ds \\ &\quad + \int_0^t\int_{\mathbb{R}} \ln(1-\theta(s,z))\tilde{N}(ds,dz) \\ &\quad + \int_0^t\int_{\mathbb{R}}\{\ln(1-\theta(s,z))+\theta(s,z)\}\nu(dz)ds\Bigg)\end{aligned}$$

is given by the equations:

$$\ln(1-\theta(t,z)) = (\exp(-\bar{\gamma}(t,T,z)-1)\frac{u(t)}{\bar{\sigma}(t,T)}$$

and

$$\begin{aligned}u(t) &= [\bar{\alpha}(t,T) - \frac{1}{2}\bar{\sigma}^2(t,T) + \int_{\mathbb{R}}[(\exp(-\bar{\gamma}(t,T,z))-1)(\theta(t,z)-1) \\ &\quad -\bar{\gamma}(t,T,z)]\nu(dz)]/\bar{\sigma}(t,T)\end{aligned}$$

when the integrands are deterministic.

5.3 Esscher Transforms

In the next two subsections we will compute the two different Esscher transforms we described earlier, and the process we will use to change our measure is our discounted zero coupon bond $\tilde{P}(t,T)$. First we will compute the Compound Return Esscher Transformed Martingale Measure, and then we will compute the Simple Return Esscher Transformed Martingale Measure.

5.3.1 Compound Return Esscher Transformed Martingale Measure

In this section we shall look at the Compound Return Esscher Transformed Martingale Measure, and then we defined a Radon-Nikodym derivative as $\frac{\exp(\int_0^T \theta(s)dX(s))}{E[\exp(\int_0^T \theta(s)dX(s))]}$, for some process $X(s)$.

In our case we have the process $\tilde{P}(t,T)$, which we recall is given by:

$$\begin{aligned}\tilde{P}(t,T) &= P(0,T)\exp\bigg(-\int_0^t \bar{\alpha}(v,T)dv - \int_0^t \bar{\sigma}(v,T)dB(v) \\ &\quad - \int_0^t \int_{\mathbb{R}} \bar{\gamma}(v,T,z)\tilde{N}(dv,dz)\bigg)\end{aligned}$$

so we will use the process

$$\begin{aligned}X(t) &= \int_0^t f(s,s)ds - \int_0^t \bar{\alpha}(v,T)dv - \int_0^t \bar{\sigma}(v,T)dB(v) \\ &\quad - \int_0^t \int_{\mathbb{R}} \bar{\gamma}(v,T,z)\tilde{N}(dv,dz) \qquad (5.35)\end{aligned}$$

in our Esscher transform. With this process we can define our Radon-Nikodym derivative $Z^\theta(T)$ by:

$$Z^\theta(t) = \frac{dQ|\mathcal{F}_t}{dP|\mathcal{F}_t} = \frac{\exp(\int_0^t \theta(u)dX(u))}{E[\exp(\int_0^t \theta(u)dX(u)]} \qquad (5.36)$$

The first thing we need to check is that our Radon-Nikodym derivative $Z^\theta(T)$ defined earlier is a martingale, which is the same as $E[Z^\theta(t)|\mathcal{F}_s] = Z^\theta(s)$ for $s \leq t \leq T$. To check that this is true, we will simplify our process $X(t)$ so we can compute $E[\exp(\int_0^t \theta(u)dX(u)]$, which is the same as $E[\exp(\int_0^t \theta(s)(f(s,s)-\bar{\alpha}(s,T))ds - \int_0^t \theta(s)\bar{\sigma}(s,T)dB(s) - \int_0^t \int_{\mathbb{R}} \theta(s)\bar{\gamma}(s,T,z)\tilde{N}(ds,dz))]$.

What we want to assume is that all our processes θ, f, $\bar{\alpha}$, $\bar{\sigma}$ and $\bar{\gamma}$ are deterministic. Then we will get that each part of $X(t)$ is independent of each other, and we can compute the characteristic function of each integral independently of the others. We also know from earlier how to compute the characteristic function of each of these processes, which is equivalent to what we want to find. First we get:

$$E\left[\exp\left(\int_0^t \theta(s)(f(s,s)-\bar{\alpha}(s,T))ds\right)\right] = \exp\left(\int_0^t \theta(s)(f(s,s)-\bar{\alpha}(s,T))ds\right) \tag{5.37}$$

since these are just deterministic functions[5]. Next we want to compute the characteristic function of our $\bar{\sigma}$-function with respect to the Brownian motion, and since we have assumed that $\bar{\sigma}$ and θ are deterministic we have that this integral is normally distributed and our characteristic function will be:

$$E\left[\exp\left(iu(-\int_0^t \theta(v)\bar{\sigma}(v,T)dB(v))\right)\right] = \exp\left(-\frac{1}{2}u^2\int_0^t (\theta(v)\bar{\sigma}(v,T))^2 dv\right) \tag{5.38}$$

Then we only have the last integral left, which is a bit more difficult to compute, but if we define $Y(t) = -\int_0^t\int_{\mathbb{R}} \theta(v)\bar{\gamma}(v,T,z)\tilde{N}(dv,dz)$ we have seen that the characteristic function of this is

$$E\left[\exp(iuY(t)\right] = \exp\left(\int_0^t\int_{\mathbb{R}} \{e^{-iu\theta(s)\bar{\gamma}(s,T,z)} - 1 + iu\theta(s)\bar{\gamma}(s,T,z)\}\nu(dz)ds\right) \tag{5.39}$$

Recall we will need some boundedness on our function $\bar{\sigma}$ and $\bar{\gamma}$ for this to be true, what kind of boundedness we need we have defined earlier. Using all this, we can compute our expected value, if we use $u = -i$ in our characteristic functions, we will get this:

$$\begin{aligned} E\left[\exp\left(\int_0^t \theta(u)dX(u)\right)\right] = \exp\Bigg(&\int_0^t \theta(s)(f(s,s)-\bar{\alpha}(s,T))ds \\ &+\frac{1}{2}\int_0^t (\theta(v)\bar{\sigma}(v,T))^2 dv \\ &+\int_0^t\int_{\mathbb{R}} \{e^{-\theta(s)\bar{\gamma}(s,T,z)} - 1 + \theta(s)\bar{\gamma}(s,T,z)\}\nu(dz)ds\Bigg) \end{aligned} \tag{5.40}$$

[5]Note that since these are not changed by the expectation, they will cancel our by the equivalent part in the numerator, so they will not give a contribution to the Radon-Nikodym derivative

To simplify this, we will define a function $\varphi(u,s)$ like this:

$$\begin{aligned}\varphi(u,s) &= u(f(s,s)-\bar{\alpha}(s,T))+\frac{1}{2}u^2\bar{\sigma}^2(s,T)\\ &\quad+\int_{\mathbb{R}}\{e^{-u\bar{\gamma}(s,T,z)}-1+u\bar{\gamma}(s,T,z)\}\nu(dz)\end{aligned} \tag{5.41}$$

and this means we can write our expected value like this:

$$E\left[\exp\left(\int_0^t \theta(s)dX(s)\right)\right] = \exp\left(\int_0^t \varphi(\theta(s),s)ds\right) \tag{5.42}$$

Now we have found an expression for the denominator of $Z^\theta(t)$, and we can use this to check that our process $Z^\theta(t)$ is a martingale. To check this, we see that for two time points, t and s, where $0\le s\le t\le T$, that the martingale property holds:

$$\begin{aligned}E[Z^\theta(t)|\mathcal{F}_s] &= E\left[\frac{\exp(\int_0^t\theta(u)dX(u))}{E[\exp(\int_0^t\theta(u)dX(u))]}|\mathcal{F}_s\right]\\ &= \frac{1}{E[\exp(\int_0^t\theta(u)dX(u))]}E\left[\exp\left(\int_s^t\theta(u)dX(u)+\int_0^s\theta(u)dX(u)\right)|\mathcal{F}_s\right]\\ &= \exp\left(\int_0^s\theta(u)dX(u)-\int_0^t\varphi(\theta(s),s)ds\right)E\left[\exp\left(\int_s^t\theta(u)dX(u)\right)\right]\\ &= \exp\left(\int_0^s\theta(u)dX(u)-\int_0^t\varphi(\theta(u),u)du+\int_s^t\varphi(\theta(u),u)du\right)\\ &= \exp\left(\int_0^s\theta(u)dX(u)-\int_0^s\varphi(\theta(u),u)du\right)\\ &= Z^\theta(s)\end{aligned} \tag{5.43}$$

Here we have used that $\int_s^t\theta(u)dX(u)$ is independent of what happens up to time s, since $(B(t)-B(s))$ and $(N(t,dz)-N(s,dz))$ is independent of what happens up to time s, and we have used that $X(s)$ is $\mathcal{F}_s$-measurable. So if we have appropriate boundedness conditions, we will get that $Z^\theta(t)$ is a martingale.

Now we have found the characteristics of our Radon-Nikodym derivative, and the next thing we will do is to find the characteristic function of $X(t)-X(s)$, with respect to $\mathcal{F}_s$, which we will use to find conditions on θ such that $\tilde{P}(t,T)$

is a martingale under our new probability measure. Since we want to condition on $\mathcal{F}_s$, we will need Bayes formula, which says:

$$\begin{aligned} E_Q[X|\mathcal{F}_s] &= E\left[\frac{Z^\theta(T)}{E[Z^\theta(T)|\mathcal{F}_s]}X|\mathcal{F}_s\right] \\ &= E\left[\frac{Z^\theta(T)}{Z^\theta(s)}X|\mathcal{F}_s\right] \end{aligned} \tag{5.44}$$

Then we can compute the characteristic function of $V(t,s) = X(t) - X(s)$ under Q like this:

$$\begin{aligned} E_Q[\exp(ixV(t,s))|\mathcal{F}_s] &= E\left[\frac{Z^\theta(T)}{Z^\theta(s)}\exp(ix(X(t)-X(s)))|\mathcal{F}_s\right] \\ &\overset{(1)}{=} \exp\left(-\int_s^T \varphi(\theta(v),v)dv\right)\cdot \\ &\quad E\left[\exp\left(\int_s^T \theta(v)dX(v) + ix(X(t)-X(s))\right)|\mathcal{F}_s\right] \\ &\overset{(2)}{=} \exp\left(-\int_s^T \varphi(\theta(v),v)dv\right) E\left[\exp\left(\int_t^T \theta(v)dX(v)\right)\cdot\right. \\ &\quad \left.\exp\left(\int_s^t (\theta(v)+ix)dX(v)\right)|\mathcal{F}_s\right] \\ &\overset{(3)}{=} \exp\left(-\int_s^T \varphi(\theta(v),v)dv\right)\exp\left(\int_t^T \varphi(\theta(v),v)dv\right)\cdot \\ &\quad \exp\left(\int_s^t \varphi(ix+\theta(v),v)dv\right) \\ &\overset{(4)}{=} \exp\left(\int_s^t \varphi(ix+\theta(v),v) - \varphi(\theta(v),v)dv\right) \end{aligned} \tag{5.45}$$

(1) Here we take out what is deterministic from $Z^\theta(T)/Z^\theta(s)$.
(2) Here we separate our integral over $[s,T]$ into two independent integrals, and we write $X(t) - X(s)$ as $\int_s^t dX(v)$.
(3) Here we compute the expected value of each integral separately, which can be done since the integrals are independent.
(4) At last we see this is the same as one integral over $[s,t]$.

From these calculations we get all we need to know about our process $X(t)$ under the Esscher transform.The next thing we will do is to find a function $\theta(v)$ such that our discounted bond price $\tilde{P}(t,T) = P(t,T)/\beta(t) =$

$P(0,T)\exp(X(t))/\beta(t)$ is a martingale under our new measure. This will mean that $E_Q[\tilde{P}(t,T)|\mathcal{F}_s] = \tilde{P}(s,T)$, and to show this, we will assume that $P(0,T)$ and $f(s,s)$ are deterministic, which will give us this:

$$\begin{aligned} E_Q[\tilde{P}(t,T)|\mathcal{F}_s] &= \frac{P(0,T)}{\beta(t)} E_Q[\exp(X(t)-X(s)+X(s))|\mathcal{F}_s] \\ &= \frac{P(0,T)}{\beta(t)} \exp(X(s)) \cdot \exp\left(\int_s^t \varphi(1+\theta(v),v) - \varphi(\theta(v),v)dv\right) \end{aligned} \tag{5.46}$$

Since we want $\tilde{P}(t,T)$ to be a martingale under Q, we see that this expected value need to be equal to $\frac{P(0,T)}{\beta(s)}X(s)$, which means we get this equation:

$$\frac{\exp\left(\int_s^t \varphi(1+\theta(v),v) - \varphi(\theta(v),v)dv\right)}{\beta(t)} = \frac{1}{\beta(s)} \tag{5.47}$$

or equivalently:

$$\int_s^t \varphi(1+\theta(v),v) - \varphi(\theta(v),v)dv = \int_s^t f(v,v)dv \tag{5.48}$$

now we can insert that

$$\begin{aligned} \varphi(u,s) &= u(f(s,s)-\bar{\alpha}(s,T)) + \frac{1}{2}u^2\bar{\sigma}^2(s,T) \\ &\quad + \int_{\mathbb{R}} \{e^{-u\bar{\gamma}(s,T,z)} - 1 + u\bar{\gamma}(s,T,z\}\nu(dz) \end{aligned}$$

which will give us this equation:

$$\begin{aligned} \int_s^t f(v,v)dv &= \int_s^t (1+\theta(v))(f(v,v)-\bar{\alpha}(v,T)) + \frac{1}{2}(1+\theta(v))^2\bar{\sigma}^2(v,T) \\ &\quad + \int_{\mathbb{R}} \{e^{-(1+\theta(v))\bar{\gamma}(v,T,z)} - 1 + (1+\theta(v))\bar{\gamma}(v,T,z)\}\nu(dz) \\ &\quad - \Bigg(\theta(v))(f(v,v)-\bar{\alpha}(v,T)) - \frac{1}{2}(\theta(v))^2\bar{\sigma}^2(v,T) \\ &\quad + \int_{\mathbb{R}} \{e^{-\theta(v)\bar{\gamma}(v,T,z)} - 1 + \theta(v)\bar{\gamma}(v,T,z)\}\nu(dz)\Bigg) dv \\ &= \int_s^t \Bigg[f(v,v) - \bar{\alpha}(v,T) + \frac{1}{2}(1+2\theta(v))\bar{\sigma}^2(v,T) \\ &\quad + \int_{\mathbb{R}} \left\{(e^{-\theta(v)\bar{\gamma}(v,T,z)})(e^{-\bar{\gamma}(v,T,z)}-1) + \bar{\gamma}(v,T,z)\right\}\nu(dz)\Bigg] dv \end{aligned} \tag{5.49}$$

As we see here, we have a $f(v,v)$ term on both sides of the equation, so this will cancel out. Since this equation should hold for all $s,t \in [0,T]$, and this means we can look at the equation point-wise so we will get this equation instead:

$$\begin{aligned} & \bar{\alpha}(v,T) - \frac{1}{2}\bar{\sigma}^2(v,T) - \int_{\mathbb{R}} \bar{\gamma}(v,T,z)\nu(dz) \\ = \quad & \theta(v)\bar{\sigma}^2(v,T) + \int_{\mathbb{R}} \left\{ (e^{-\theta(v)\bar{\gamma}(v,T,z)})(e^{-\bar{\gamma}(v,T,z)} - 1) \right\} \nu(dz) \end{aligned} \tag{5.50}$$

To show that this equation has a solution, we define the function $f(\theta)$ by:

$$f(\theta) = \theta\bar{\sigma}^2(v,T) + \int_{\mathbb{R}} \left\{ (e^{-\theta\bar{\gamma}(v,T,z)})(e^{-\bar{\gamma}(v,T,z)} - 1) \right\} \nu(dz) \tag{5.51}$$

and this has derivative equal to:

$$f'(\theta) = \bar{\sigma}^2(v,T) + \int_{\mathbb{R}} \left\{ (-\bar{\gamma}(v,T,z)e^{-\theta\bar{\gamma}(v,T,z)})(e^{-\bar{\gamma}(v,T,z)} - 1) \right\} \nu(dz) \tag{5.52}$$

and since $-\bar{\gamma}(v,T,z) < 0 \Rightarrow e^{-\bar{\gamma}(v,T,z)} - 1 < 0$, and $-\bar{\gamma}(v,T,z) > 0 \Rightarrow e^{-\bar{\gamma}(v,T,z)} - 1 > 0$, and obviously $\bar{\sigma}^2(v,T) > 0$, we get that our derivative is positive for all θ, which means $f : \mathbb{R} \to \mathbb{R}$ is strictly increasing. So if $f(\theta)$ is continuous, and satisfy this inequality:

$$\lim_{\theta\to-\infty} f(\theta) < \bar{\alpha}(v,T) - \frac{1}{2}\bar{\sigma}^2(v,T) - \int_{\mathbb{R}} \bar{\gamma}(v,T,z)\nu(dz) < \lim_{\theta\to\infty} f(\theta) \tag{5.53}$$

we will have a unique solution to our equation. This inequality will hold so long $|\bar{\alpha}(v,T) - \frac{1}{2}\bar{\sigma}^2(v,T) - \int_{\mathbb{R}} \bar{\gamma}(v,T,z)\nu(dz)| < \infty$, since the linear term in $f(\theta)$ will make $\lim_{\theta\to-\infty} f(\theta) = -\infty$, and $\lim_{\theta\to\infty} f(\theta) = \infty$.

5.3.2 Simple Return Esscher Transformed Martingale Measure

In this section we will look at the Simple return Esscher transformed martingale measure, and we recall that then we again had a process of the form $S(t) = S(0)e^{Y(t)}$, but instead of using the process $Y(t)$ to define the Radon-Nikodym derivative, we will look at

$$dS(t) = S(t^-)dZ(t) \tag{5.54}$$

and use $Z(t)$ to change our measure. All the computations in this section will be similar to what we did in the last section, but we will end up with different expressions. A lot of the computations could possibly be skipped since we have the same results in the last section, but they will be included to show the results.

Again we are interested in finding a measure such that

$$\begin{aligned}\tilde{P}(t,T) &= P(0,T)\exp\Big(-\int_0^t \bar{\alpha}(v,T)dv - \int_0^t \bar{\sigma}(v,T)dB(v) \\ &\quad - \int_0^t\int_{\mathbb{R}} \bar{\gamma}(v,T,z)\tilde{N}(dv,dz)\Big)\end{aligned}$$

is a martingale. To find this, we recall the dynamics of $P(t,T)$ is given by:

$$\begin{aligned}dP(t,T) &= P(t,T)\left(f(t,t) - \bar{\alpha}(t,T) + \frac{1}{2}\bar{\sigma}^2(t,T)\right)dt - P(t,T)\bar{\sigma}(t,T)dB(t) \\ &\quad + \int_{\mathbb{R}} P(t,T)\left[\exp(-\bar{\gamma}(t,T,z)) - 1 + \bar{\gamma}(t,T,z)\right]\nu(dz)dt \\ &\quad + \int_{\mathbb{R}} P(t^-,T)\left[\exp(-\bar{\gamma}(t,T,z)) - 1\right]\tilde{N}(dt,dz) \qquad (5.55)\end{aligned}$$

So the process we will use to define our Radon-Nikodym derivative is:

$$\begin{aligned}Y(t) &= \int_0^t \left(f(s,s) - \bar{\alpha}(s,T) + \frac{1}{2}\bar{\sigma}^2(s,T)\right)ds - \int_0^t \bar{\sigma}(s,T)dB(s) \\ &\quad + \int_0^t\int_{\mathbb{R}} \left[\exp(-\bar{\gamma}(s,T,z)) - 1 + \bar{\gamma}(s,T,z)\right]\nu(dz)ds \\ &\quad + \int_0^t\int_{\mathbb{R}} \left[\exp(-\bar{\gamma}(s,T,z)) - 1\right]\tilde{N}(ds,dz) \qquad (5.56)\end{aligned}$$

and as before we need a time dependence in our θ parameter, so we want to look at $\int_0^t \theta(s)dY(s)$ instead of $\theta Y(t)$. So as before our Radon-Nikodym derivative will be given by

$$Z^\theta(T) = \frac{\exp(\int_0^T \theta(s)dY(s))}{E[\exp(\int_0^T \theta(s)dY(s))]} \qquad (5.57)$$

First we shall compute $E[e^{\int_0^t \theta(s)dY(s)}]$, and to compute this we need deterministic and nicely bounded functions, and if that is satisfied, we get this:

$$
\begin{aligned}
E[e^{\int_0^t \theta(s)dY(s)}] &= \exp\Bigg(\int_0^t \theta(s)\big[f(s,s)-\bar{\alpha}(s,T) \\
&\quad +\frac{1}{2}(1+\theta(s))\bar{\sigma}^2(s,T)\big]ds \\
&\quad +\int_0^t\int_{\mathbb{R}}\big[e^{\theta(s)(\exp(-\bar{\gamma}(s,T,z))-1)} \\
&\quad -1+\theta(s)\bar{\gamma}(s,T,z)\big]\nu(dz)ds\Bigg) \\
&= \exp\left(\int_0^t \varphi(\theta(s),s)ds\right) \qquad (5.58)
\end{aligned}
$$

Here φ is defined as $\varphi(u,s) = u\left(f(s,s)-\bar{\alpha}(s,T)+\frac{1}{2}(1+u)\bar{\sigma}^2(s,T)\right)$ $+\int_{\mathbb{R}}[e^{u(\exp(-\bar{\gamma}(s,T,z))-1)}-1+u\bar{\gamma}(s,T,z)]\nu(dz)$. When we have this, we can check that $Z^\theta(t)$ is a martingale, so we see that for two time-points, t,s, where $0\le s\le t\le T$, we have that

$$
\begin{aligned}
E[Z^\theta(t)|\mathcal{F}_s] &= E[e^{\int_0^t \theta(u)dY(u)-\int_0^t \varphi(\theta(u),u)du}|\mathcal{F}_s] \\
&= e^{\int_0^s \theta(u)dY(u)-\int_0^s \varphi(\theta(u),u)du} \\
&= Z^\theta(s) \qquad (5.59)
\end{aligned}
$$

This computation is done in exactly the same manner as the corresponding one in the last section, so look at that for details. The next thing we will do, is to compute the characteristic function of $V(t,s)=X(t)-X(s)$ under our new probability measure Q, which will be done in the same manner as in the previous section.

$$
\begin{aligned}
E_Q[\exp(ixV(t,s))|\mathcal{F}_s] &= E\left[\frac{Z^\theta(T)}{Z^\theta(s)}\exp(ix(X(t)-X(s)))|\mathcal{F}_s\right] \\
&= e^{-\int_s^T \varphi(\theta(u),u)du} E\left[\exp\left(\int_s^T \theta(u)dY(u) + \int_s^t ixdY(u)\right)\right] \\
&= e^{-\int_s^t \varphi(\theta(u),u)du} E\left[\exp\left(\int_s^t \theta(u)dY(u) + \int_s^t ixdX(u)\right)\right] \\
&= e^{-\int_s^t \varphi(\theta(u),u)du} E\Bigg[\exp\Bigg(\int_s^t \Big[\theta(u)\left(f(u,u)-\bar{\alpha}(u,T)\right. \\
&\quad \left.+\frac{1}{2}\bar{\sigma}^2(u,T)\right) + ix(f(u,u)-\bar{\alpha}(u,T))\Big]du \\
&\quad -\int_s^t (\theta(u)+ix)\bar{\sigma}(u,T)dB(u) \\
&\quad +\int_s^t\int_{\mathbb{R}} \theta(u)\left[\exp(-\bar{\gamma}(u,T,z))-1+\bar{\gamma}(u,T,z)\right]\nu(dz)du \\
&\quad +\int_s^t\int_{\mathbb{R}} \{\theta(u)\left(\exp(-\bar{\gamma}(u,T,z))-1\right)-ix\bar{\gamma}(u,T,z)\}\tilde{N}(du,dz)\Bigg)\Bigg]
\end{aligned}
\tag{5.60}
$$

We will simplify this by calling the expression in the exponent of (5.60) for $V(t)$, then we can calculate $E[\exp(V(t))]$:

$$
\begin{aligned}
E[\exp(V(t))] &= \exp\Bigg(\int_s^t \Big[\theta(u)\Big(f(u,u)-\bar{\alpha}(u,T)+\frac{1}{2}\bar{\sigma}^2(u,T)\Big) \\
&\quad +ix(f(u,u)-\bar{\alpha}(u,T)) + \int_{\mathbb{R}} \theta(u)[\exp(-\bar{\gamma}(u,T,z)) \\
&\quad -1+\bar{\gamma}(u,T,z)]\nu(dz)\Big]du + \frac{1}{2}\int_s^t ((\theta(u)+ix)\bar{\sigma}(u,T))^2 du \\
&\quad +\int_s^t\int_{\mathbb{R}} (e^{\theta(u)[\exp(-\bar{\gamma}(u,T,z))-1]-ix\bar{\gamma}(u,T,z)}-1 \\
&\quad -\theta(u)[\exp(-\bar{\gamma}(u,T,z))-1]+ix\bar{\gamma}(u,T,z))\nu(dz)du\Bigg)
\end{aligned}
\tag{5.61}
$$

As in the last section, our next step will be to find a function $\theta(v)$, such that our discounted bond price, $\tilde{P}(t,T) = \frac{P(t,T)}{\beta(t)}$ is a martingale under this

Esscher transform, and this means that

$$\begin{aligned}
E_Q[\tilde{P}(t,T)|\mathcal{F}_s] &= E_Q\left[\frac{P(0,T)\exp(X(t))}{\beta(t)}|\mathcal{F}_s\right] \\
&= \frac{P(0,T)\exp(X(s)+\int_t^T \varphi(\theta(u),u)du)}{\beta(t)} E_Q\left[\exp(X(t)-X(s))|\mathcal{F}_s\right] \\
&= \frac{P(0,T)\exp(X(s)-\int_s^t \varphi(\theta(u),u)du)}{\beta(t)} E[\exp(X(t)-X(s))] \\
&= \frac{P(0,T)\exp(X(s)-\int_s^t \varphi(\theta(u),u)du)}{\beta(t)} E[\exp(V(t))]|_{ix=1} \qquad (5.62)
\end{aligned}$$

should be equal to $\frac{P(0,T)\exp(X(s))}{\beta(s)}$. For this to be true, we get an equation in $\theta(t)$ that needs to hold, namely $\exp(-\int_s^t \varphi(\theta(u),u)du)E[\exp(V(t))]|_{ix=1} = \exp(\int_s^t f(u,u)du)$. This will be solved in the same way as in the previous section, we insert from equation (5.61), and our expression for $\varphi(u,s)$, and after some simplifications we get this equation:

$$\begin{aligned}
f(u,u) &= f(u,u) - \bar{\alpha}(u,T) + (\theta(u)+\frac{1}{2})\bar{\sigma}^2(u,T) \\
&\quad + \int_{\mathbb{R}} \left(e^{\theta(u)(\exp(-\bar{\gamma}(u,T,z))-1)}(e^{-\bar{\gamma}(u,T,z)}-1) + \bar{\gamma}(u,T,z)\right)\nu(dz)
\end{aligned} \qquad (5.63)$$

and we can rewrite this equation like this:

$$\begin{aligned}
&\bar{\alpha}(u,T) - \frac{1}{2}\bar{\sigma}^2(u,T) - \int_{\mathbb{R}} \bar{\gamma}(u,T,z)\nu(dz) \\
&= \theta(u)\bar{\sigma}^2(u,T) + \int_{\mathbb{R}} e^{\theta(u)(\exp(-\bar{\gamma}(u,T,z))-1)}(e^{-\bar{\gamma}(u,T,z)}-1)\nu(dz)
\end{aligned} \qquad (5.64)$$

To show that this equation has a solution, we will follow in the same manner as in the previous section, namely define a function $f(\theta)$ as:

$$f(\theta) = \theta\bar{\sigma}^2(u,T) + \int_{\mathbb{R}} e^{\theta(\exp(-\bar{\gamma}(u,T,z))-1)}(e^{-\bar{\gamma}(u,T,z)}-1)\nu(dz) \qquad (5.65)$$

and the derivate of f is:

$$f'(\theta) = \bar{\sigma}^2(u,T) + \int_{\mathbb{R}} e^{\theta(\exp(-\bar{\gamma}(u,T,z))-1)}(e^{-\bar{\gamma}(u,T,z)}-1)^2\nu(dz) \qquad (5.66)$$

and it is clear that $f'(\theta) > 0$. Then we have a function which is strictly increasing, and if $\bar{\sigma}(u,T) \neq 0$ it is clear that $\lim_{\theta\to\pm\infty} f(\theta) = \pm\infty$, so boundedness of our other functions implies that we have a unique solution.

At last we will compare this solution to the solution we got in the MEMM-case. Here we had two equations which gave us our probability measure, namely:

$$\begin{aligned} &\bar{\alpha}(t,T) - \frac{1}{2}\bar{\sigma}^2(t,T) - \int_{\mathbb{R}} \bar{\gamma}(t,T,z)\nu(dz) \\ &= \bar{\sigma}(t,T)u(t) + \int_{\mathbb{R}} (1-\theta(t,z))[\exp(-\bar{\gamma}(t,T,z)) - 1]\nu(dz) \end{aligned}$$

and

$$\ln(1-\theta(t,z)) = (\exp(-\bar{\gamma}(t,T+\delta,z)) - 1)\frac{u(t)}{\bar{\sigma}(t,T+\delta)}$$

If we set $u(t) = \bar{\sigma}(u,T)v(t)$, and we solve for $\theta(t,z)$ in these equations, we get:

$$\begin{aligned} &\bar{\alpha}(t,T) - \frac{1}{2}\bar{\sigma}^2(t,T) - \int_{\mathbb{R}} \bar{\gamma}(t,T,z)\nu(dz) \\ = \quad & v(t)\bar{\sigma}^2(t,T) + \int_{\mathbb{R}} e^{v(t)(\exp(-\bar{\gamma}(t,T,z))-1)}(e^{-\bar{\gamma}(t,T,z)} - 1)\nu(dz) \end{aligned}$$

and we see that this equation is the same as the equation we got earlier in this section, and this means that the MEMM and the Simple Return Esscher Transformed Martingale Measure are the same.

5.4 The Price of a European Call Option

In the previous sections we found different ELMM's for $\tilde{P}(t,T)$, and in this section we will show how these measures could be used to find arbitrage free prices for a European Call Option. As we have seen earlier, we define our option by a function $H(x)$, and then our option price will be given by:

$$P_Q(L,H) = E_Q\left[\exp\left(-\int_0^{T+\delta} r(s)ds\right) H(L(T,T))\right] \tag{5.67}$$

To compute this, we will assume our interest rate $r(s)$ is deterministic, so we can move this out of the expectation. What we are left with computing then is expectations of the form $E[H(X(T))]$.

As we saw earlier, if H and $\hat{H}$ are in $L^1(\mathbb{R})$, then we can use Fourier transforms to compute $E[H(X(T))]$, like this:

$$\begin{aligned} E[H(X(T))] &= \frac{1}{2\pi} E\left[\int_{\mathbb{R}} \hat{H}(y) e^{iyX(T)} dy\right] \\ &= \frac{1}{2\pi} \int_{\mathbb{R}} \hat{H}(y) E\left[e^{iyX(T)}\right] dy \end{aligned} \tag{5.68}$$

In the second line, we move the expectation inside the integral, and for this to work, we need that the conditions of the Fubini theorem holds. We also need that $E[\exp(iyX(T))]$ exists, and we need to be able to compute this for us to get an expression for the option price, but this is the same as the characteristic function of $X(T)$.

So to compute the option price, we need two things, first we need to find the Fourier transform of the function H, and we need to find the characteristic function of $X(T)$.

To find the function we will Fourier transform, we will first look at the characteristic function. We are interested in a process of the form $L(T,T) = \frac{P(0,T)}{\delta P(0,T+\delta)} \exp(Z(T)) - \frac{1}{\delta}$, for some process $Z(T)$. Instead of finding the characteristic function of this, we will see it is more convenient to look at the characteristic function of $Z(T)$ in some cases. Since we are interested at the characteristic function of $Z(T)$, instead of $L(T,T)$, we need that the function we Fourier transform, is a function of $Z(T)$ as well, so we want a function $w(x)$, where $w(Z(T)) = H(L(T,T))$.

Now we will show how our function $w(x)$ will be defined when we work with a European Call Option. Recall that the European Call Option pays the difference between $L(T,T)$ and a cap rate K, whenever $L(T,T) > K$, mathematically this is $(L(T,T) - K)^+$. Since we are interested in a function of the process $X(T)$, where $X(T)$ is the exponent of $L(T,T)$, we will insert our expression for $L(T,T)$, and see how to separate $X(T)$.

$$\begin{aligned}
(L(T,T)-K)^+ &= \left(\frac{P(0,T)}{\delta P(0,T+\delta)}\exp(X(T)) - \frac{1}{\delta} - K\right)^+ \\
&= \left(\frac{P(0,T)}{\delta P(0,T+\delta)}\left(\exp(X(T)) - \frac{\delta P(0,T+\delta)}{P(0,T)}(\frac{1}{\delta}+K)\right)\right)^+ \\
&= \frac{P(0,T)}{\delta P(0,T+\delta)}\left(\exp(X(T)) - \frac{\delta P(0,T+\delta)}{P(0,T)}(\frac{1}{\delta}+K)\right)^+ \\
&= \frac{P(0,T)}{\delta P(0,T+\delta)}(\exp(X(T))-K')^+ \qquad (5.69)
\end{aligned}$$

here $K' = \frac{\delta P(0,T+\delta)}{P(0,T)}(\frac{1}{\delta}+K)$. So if we assume that $\frac{P(0,T)}{\delta P(0,T+\delta)}$ is deterministic, we can move this out of the expectation, and what we are left with is a function of the form $w(x) = (\exp(x)-K)^+$, and this is what we shall use the Fourier transform on.

$$\begin{aligned}
\hat{w}(z) &= \int_{-\infty}^{\infty} \exp(izx)w(x)dx \\
&= \int_{-\infty}^{\infty} \exp(izx)(\exp(x)-K)^+dx \\
&\overset{(1)}{=} \int_{\ln K}^{\infty} \exp(izx)(\exp(x)-K)dx \\
&= \int_{\ln K}^{\infty} (\exp(x(iz+1)-K\exp(izx))dx \\
&\overset{(2)}{=} \left[\frac{1}{iz+1}\exp(x(iz+1)) - K\frac{1}{iz}\exp(izx)\right]_{x=\ln K}^{x=\infty} \\
&= \left[\left(\frac{1}{iz+1}\exp(x) - K\frac{1}{iz}\right)\exp(izx)\right]_{x=\ln K}^{x=\infty} \\
&\overset{(3)}{=} \frac{-K^{iz+1}}{iz-z^2} \qquad (5.70)
\end{aligned}$$

(1) Here we have used that $(\exp(x)-K)^+ \neq 0$ when $x > \ln(K)$.
(2) Here we have done the integration, but not inserted the limits.
(3) Here we insert the limits, and calculate what we get. For the value to exist at $x=\infty$, we need som conditions on z, if we set $z = a+ib$, we get $\exp(x(1+iz)) = \exp(x(1-b+ia))$, and if $b<1$, then $|\exp(x(1-b+ia))|_{x=\infty}| = \infty$, and if $b=1$ we get $\exp(x(ia))$, which is not given for $x=\infty$, so we need $b = \mathrm{im}(z) > 1$ for our integral to be well defined.

Now we have our Fourier transformed function $\hat{w}(x)$, so what we need is our characteristic function $E_Q[\exp(iyX(T))]$, and from our definition of the LIBOR rates (4.20), we get this expression for $X(T)$:

$$\begin{aligned} X(T) &= \int_0^T \bar{\alpha}(v, T+\delta) - \bar{\alpha}(v, T)dv + \int_0^T \bar{\sigma}(v, T+\delta) - \bar{\sigma}(v, T)dB(v) \\ &\quad + \int_0^T \int_{\mathbb{R}} \bar{\gamma}(v, T+\delta, z) - \bar{\gamma}(v, T, z)\tilde{N}(dv, dz) \end{aligned} \tag{5.71}$$

We also see that when we shall compute $E_Q[\exp(iyX(T))] = E[Z(T)\exp(iyX(T))]$, we will get something different for each of our three measures, since the process $Z(T)$ will change. Now we will compute $E_Q[\exp(iyX(T))]$, for a general measure change. First we recall that

$$\begin{aligned} Z(t) &:= \exp\Big(- \int_0^t u(s)dB(s) - \frac{1}{2}\int_0^t u^2(s)ds \\ &\quad + \int_0^t \int_{\mathbb{R}} \ln(1-\theta(s,z))\tilde{N}(ds,dz) \\ &\quad + \int_0^t \int_{\mathbb{R}} [\ln(1-\theta(s,z)) + \theta(s,z)]\nu(dz)ds \Big) \end{aligned}$$

and if it shall be possible for us to compute the characteristic function, we need that $\bar{\alpha}$, $\bar{\sigma}$ and $\bar{\gamma}$ are deterministic, this will also imply that u and θ are deterministic. Then we get by a straight forward computation that:

$$
\begin{aligned}
E_Q[\exp(iuX(T))] &= E\Bigg[\exp\Bigg(\int_0^T \{iu(\bar{\alpha}(v,T+\delta)-\bar{\alpha}(v,T))-\frac{1}{2}u^2(v)\}dv \\
&\quad + \int_0^T \{iu(\bar{\sigma}(v,T+\delta)-\bar{\sigma}(v,T))-u(v)\}dB(v) \\
&\quad + \int_0^T\int_{\mathbb{R}} \{\ln(1-\theta(v,z))+\theta(v,z)\}\nu(dz)dv \\
&\quad + \int_0^T\int_{\mathbb{R}} \{iu(\bar{\gamma}(v,T+\delta,z)-\bar{\gamma}(v,T,z)) \\
&\quad + \ln(1-\theta(s,z))\}\tilde{N}(dv,dz)\Bigg)\Bigg] \\
&= \exp\Bigg(\int_0^T \{iu(\bar{\alpha}(v,T+\delta)-\bar{\alpha}(v,T))-\frac{1}{2}u^2(v)\}dv \\
&\quad + \frac{1}{2}\int_0^T \{iu(\bar{\sigma}(v,T+\delta)-\bar{\sigma}(v,T))-u(v)\}^2dv \\
&\quad + \int_0^T\int_{\mathbb{R}} \{\ln(1-\theta(v,z))+\theta(v,z)\}\nu(dz)dv \\
&\quad + \int_0^T\int_{\mathbb{R}} \{e^{iu(\bar{\gamma}(v,T+\delta,z)-\bar{\gamma}(v,T,z))+\ln(1-\theta(s,z))}-1 \\
&\quad - (iu(\bar{\gamma}(v,T+\delta,z)-\bar{\gamma}(v,T,z)) \\
&\quad + \ln(1-\theta(s,z)))\}\nu(dz)dv\Bigg)
\end{aligned} \tag{5.72}
$$

Then we have an expression for our characteristic function and we have our Fourier transformed function $\hat{w}$, and we can combine this to get our final expression. If we combine (5.67), (5.68) and (5.70) we will get this formula for our option price:

$$
\begin{aligned}
P_Q(L,H) &= E_Q\left[\exp\left(-\int_0^{T+\delta} r(s)ds\right)H(L(T,T))\right] \\
&= \frac{1}{2\pi}\frac{P(0,T)}{\delta P(0,T+\delta)}\int_{-\infty+ai}^{+\infty+ai}\frac{-(K')^{iy+1}}{iy-y^2}E_Q[\exp(iyX(T))]dy
\end{aligned} \tag{5.73}
$$

and to get our final result we will insert (5.72), and that $K' = \frac{\delta P(0,T+\delta)}{P(0,T)}(\frac{1}{\delta}+K)$ as well, we also recall that we need $a > 1$ for the Fourier transform to exist.

5.5 Investment Strategies

In the next two subsections we will see how to find investment strategies for our HJM model when we want to maximize the expected utility from our investment. We will first use the duality method to find our investment strategy when we have an exponential utility function, and then we will find the same strategy by using the maximum principle directly. The reason for why we do two computations that give the same result that the duality method is proven to work when the utility function satisfy the Inada conditions, something our utility function will not satisfy, so we will show these are not always necessary.

5.5.1 Finding Investment Strategies by the Duality Approach

In this section we look at the discounted zero coupon bond $\tilde{P}(t,T)$, and we want to find an investment strategy that maximizes the expected utility when we invest in this bond. Our approach for solving this problem is first to use the duality method described earlier.

In our case the discounted risky asset is $\tilde{P}(t,T)$, and we have a self-financing investment strategy $\pi(t)$. Our portfolio value at time t, if we start with the amount x, is then given by:

$$X_x^{\pi}(t) = x + \int_0^t \pi(s^-)dS(s);\ 0 \leq t \leq T \tag{5.74}$$

The problem we then want to solve is finding an investment strategy $\pi^* \in \mathcal{A}$, such that

$$u(x) = \sup_{\pi \in \mathcal{A}} E[U(X_x^{\pi}(T))] = E[U(X_x^{\pi^*}(T))] \tag{5.75}$$

where $\mathcal{A}$ is the set of admissible investment strategies, for an utility function U. Recall that the dual problem to this is to find $Q^* \in \mathcal{M}$ such that

$$v(y) = \inf_{Q \in \mathcal{M}} E\left[V\left(y\frac{dQ}{dP}\right)\right] = E\left[V\left(y\frac{dQ^*}{dP}\right)\right] \tag{5.76}$$

where $\mathcal{M}$ is the set of ELMM's

$$V(y) = \sup_{x>0}\{U(x) - xy\};\ y > 0 \tag{5.77}$$

and

$$U(x) = \inf_{y>0}\{V(y) + xy\};\ x > 0 \tag{5.78}$$

then we had these results:

$$X_x^{\pi^*}(T) = -V'\left(y\frac{dQ^*}{dP}\right);\ x = -v'(y) \tag{5.79}$$

$$y\frac{dQ^*}{dP} = U'(X_x^{\pi^*}(T));\ y = u'(x) \tag{5.80}$$

So if we can find Q^*, for a function V, then we also find an optimal investment strategy π^*, and optimal wealth $X_x^{\pi^*}(T)$ for an utility function U. Then we see that problems of the form (5.76) is the same as the problems we solved when we found our minimal quadratic distance and the minimal entropy martingale measure. In that section we looked at the functions $V_1(x) = x^2$ and $V_2(x) = x\ln(x)$, and if we start looking at $V_2(x)$, we get:

$$U(x) = \inf_{y>0}\{y\ln(y) + xy\} \tag{5.81}$$

If we differentiate $y\ln(y) + xy$ with respect to y, and set this equal to 0, we get $1 + \ln(y) + x = 0 \Rightarrow y = e^{-x-1}$, if we insert this into (5.81) we will get $U(x) = -e^{-x-1}$. So finding

$$\inf_{Q\in\mathcal{M}} E\left[\frac{dQ}{dP}\ln\left(\frac{dQ}{dP}\right)\right] \tag{5.82}$$

is the same as finding

$$\sup_{\pi\in\mathcal{A}} E\left[-\exp(-X_x^\pi(T) - 1)\right] \tag{5.83}$$

and we can see that if we take the function $U_2(x) = 1 + e^1U(x) = (1 - \exp(-x))$, which is in the class of exponential utility functions, we get that solving our original problem is the same as solving the investment problem for the function $U_2(x)$, since the difference between U and U_2 is adding and multiplication of a constant. This means that a π that maximizes (5.83), will also maximize

$$\sup_{\pi\in\mathcal{A}} E\left[U_2(X_x^\pi(T)))\right] \tag{5.84}$$

If we had done the same thing for $V_1(x)$, we would have gotten $U(x) = -\frac{1}{4}x^2$, which is not that much used as an utility function, so we will not use this. Now we have seen what kind of utility function we get, and we can start finding our optimal investment strategy. First we recall that $X_x^\pi(t)$,

$0 \leq t \leq T$, is given by:

$$\begin{aligned}
X_x^\pi(t) &= x + \int_0^t \pi(s) d\tilde{P}(s,T) \\
&= x + \int_0^t \pi(s)\tilde{P}(s,T)(-\bar{\alpha}(s,T) + \frac{1}{2}\bar{\sigma}^2(s,T))ds \\
&\quad - \int_0^t \pi(s)\tilde{P}(s,T)\bar{\sigma}(s,T)dB(s) + \int_0^t \int_{\mathbb{R}} \pi(s)\tilde{P}(s,T)[\exp(-\bar{\gamma}(s,T,z)) - 1 \\
&\quad + \bar{\gamma}(s,T,z)]\nu(dz)ds + \int_0^t \pi(s^-)\tilde{P}(s^-,T)[\exp(-\bar{\gamma}(s,T,z)) - 1]\tilde{N}(ds,dz)
\end{aligned} \tag{5.85}$$

and from (5.79) we get that

$$\begin{aligned}
X_x^{\pi^*}(T) &= -1 - \ln(y) - \ln\left(\frac{dQ^*}{dP}\right) \\
&= -1 - \ln(y) + \int_0^T u(s)dB(s) + \frac{1}{2}\int_0^T u^2(s)ds \\
&\quad - \int_0^T \int_{\mathbb{R}} \ln(1-\theta(s,z))\tilde{N}(ds,dz) \\
&\quad - \int_0^T \int_{\mathbb{R}} (\ln(1-\theta(s,z)) + \theta(s,z))\nu(dz)ds
\end{aligned} \tag{5.86}$$

So we have the dynamics of the process, with unknown variable $\pi(t)$ and we have the optimal endpoint, so this is a BSDE. We also know from the section about the MEMM that $u(t)$ and $\theta(t,z)$ solves these equations:

$$\ln(1-\theta(t,z)) = (\exp(-\bar{\gamma}(t,T,z)) - 1)\frac{u(t)}{\bar{\sigma}(t,T)} \tag{5.87}$$

and

$$\begin{aligned}
&\bar{\alpha}(t,T) - \frac{1}{2}\bar{\sigma}^2(t,T) - \int_{\mathbb{R}} \bar{\gamma}(t,T,z)\nu(dz) \\
&= \bar{\sigma}(t,T)u(t) + \int_{\mathbb{R}} (1-\theta(t,z))[\exp(-\bar{\gamma}(t,T,z)) - 1]\nu(dz)
\end{aligned} \tag{5.88}$$

Since this BSDE has a quite similar endpoint as the one we got for the MEMM, it is natural to guess on a solution that is quite similar. We try with $X_x^{\pi^*}(t) = x - p(t) = -\ln(yG(t)) - \phi_t$, and we have that $x = -\ln(y) - 1$, then we get:

$$\begin{aligned} X_x^{\pi^*}(t) &= -\ln(y) + \int_0^t u(s)dB(s) + \frac{1}{2}\int_0^t u^2(s)ds - \int_0^t \int_{\mathbb{R}} \ln(1-\theta(s,z))\tilde{N}(ds,dz) \\ &\quad - \int_0^t \int_{\mathbb{R}} \{\ln(1-\theta(s,z)) + \theta(s,z)\}\nu(dz)ds - \int_0^t \phi_s' ds \end{aligned} \tag{5.89}$$

If we compare (5.85) and (5.89), we will get this expression for $\pi^*(t)$:

$$\pi^*(t) = -\frac{u(t)}{\tilde{P}(t,T)\bar{\sigma}(t,T)} \tag{5.90}$$

when we look at the integral with respect to the Brownian motion. If we insert this into (5.85) we will get these equations:

$$\begin{aligned} &\frac{1}{2}u^2(t) - \int_{\mathbb{R}} (\ln(1-\theta(t,z)) + \theta(t,z))\nu(dz) - \phi_t' \\ = \; &\frac{u(t)}{\bar{\sigma}(t,T)}\{\bar{\alpha}(t,T) - \frac{1}{2}\bar{\sigma}^2(t,T)) - \int_{\mathbb{R}} [\exp(-\bar{\gamma}(t,T,z)) - 1 \\ &+\bar{\gamma}(t,T,z)]\nu(dz)\} \end{aligned} \tag{5.91}$$

and

$$\ln(1-\theta(t,z)) = \frac{u(t)}{\bar{\sigma}(t,T)}[\exp(-\bar{\gamma}(t,T,z)) - 1] \tag{5.92}$$

Since this is two equations, and we have no new variables, we need these equations to be the same as the ones we have from before, if our expression of $\pi^*(t)$ shall be a valid investment strategy. If we start comparing, we see that equation (5.92) is the same as (5.87), so this will hold, so we only need to check if (5.91) holds. If we insert from our MEMM what ϕ_t' and $\ln(1-\theta(t,z))$ is, we get:

$$\begin{aligned} &u^2(t) - \int_{\mathbb{R}} \theta(t,z)(\exp(-\bar{\gamma}(t,T,z)) - 1)\frac{u(t)}{\bar{\sigma}(t,T)}\nu(dz) \\ = \; &\frac{u(t)}{\bar{\sigma}(t,T)}\{\bar{\alpha}(t,T) - \frac{1}{2}\bar{\sigma}^2(t,T) - \int_{\mathbb{R}} [\exp(-\bar{\gamma}(t,T,z)) - 1 \\ &+\bar{\gamma}(t,T,z)]\nu(dz)\} \end{aligned} \tag{5.93}$$

If we multiply with $\frac{\bar{\sigma}(t,T)}{u(t)}$ on both sides of the equation, we see this equation is the same as:

$$\begin{aligned} &\bar{\sigma}(t,T)u(t) + \int_{\mathbb{R}} (\exp(-\bar{\gamma}(t,T,z)) - 1)(1-\theta(t,z))\nu(dz) \\ = \; &\bar{\alpha}(t,T) - \frac{1}{2}\bar{\sigma}^2(t,T) - \int_{\mathbb{R}} \bar{\gamma}(t,T,z)\nu(dz) \end{aligned} \tag{5.94}$$

but this is the same as equation (5.88), so this holds as well. So we get that

$$X_x^{\pi^*}(T) = -\ln(y) - \phi_T - \ln(G(T)) \tag{5.95}$$

At last we need to check that our initial condition holds. Our initial condition says that $x = -\ln(y) - 1$, and this shall be equal to $-\ln(y) - \phi_T$, which means that $\phi_T = 1$, and this is the same as we got when we solved the MEMM problem, so all is as it should be.

To conclude we say that our optimal portfolio for the investment problem

$$u(x) = \sup_{\pi \in \mathcal{A}} E\left[1 - e^{-X_x^{\pi}(T)}\right]$$

is

$$\pi^*(t) = -\frac{u(t)}{\bar{\sigma}(t,T)\tilde{P}(t,T)}$$

where $u(t)$ is given by (5.87) and (5.88), and the optimal final wealth will be:

$$X_x^{\pi^*}(T) = x - \ln(G(T)) \tag{5.96}$$

Then we get:

$$\begin{aligned} u(x) &= E\left[1 - e^{\ln(G(T))-x}\right] \\ &= E\left[1 - G(T)e^{-x}\right] \\ &= 1 - e^{-x} \end{aligned} \tag{5.97}$$

since $E[G(T)] = 1$, and we get that $u(x) = U_2(x)$.

5.5.2 Finding Investment Strategies by the Maximum Principle

In this section we will find the same investment strategy as in the previous section, but instead of using the duality method, we will compute it in a straight forward way, by maximizing the expected utility with regards to the utility function $U(x) = 1 - \exp(-x)$. The reason for why we do this is to show that the Inada conditions are not necessary for the duality approach to work.

Again we see our wealth process is given by:

$$\begin{aligned}
X_x^\pi(t) &= x + \int_0^t \pi(s^-) d\tilde{P}(s^-, T) \\
&= x + \int_0^t \pi(s)\tilde{P}(s,T)(-\bar{\alpha}(s,T) + \frac{1}{2}\bar{\sigma}^2(s,T))ds \\
&\quad - \int_0^t \pi(s)\tilde{P}(s,T)\bar{\sigma}(s,T)dB(s) + \int_0^t \int_{\mathbb{R}} \pi(s)\tilde{P}(s,T)[\exp(-\bar{\gamma}(s,T,z)) - 1 \\
&\quad + \bar{\gamma}(s,T,z)]\nu(dz)ds + \int_0^t \pi(s^-)\tilde{P}(s^-,T)[\exp(-\bar{\gamma}(s,T,z)) - 1]\tilde{N}(ds,dz)
\end{aligned} \tag{5.98}$$

and our Hamiltonian function will be

$$\begin{aligned}
H(t,x,\pi,p,q,r) &= x(-\bar{\alpha}(t,T) + \frac{1}{2}\bar{\sigma}^2(t,T) + \int_{\mathbb{R}} (\exp(-\bar{\gamma}(t,T,z)) \\
&\quad -1 + \bar{\gamma}(t,T,z))\nu(dz))p - x\bar{\sigma}(t,T)q \\
&\quad + \int_{\mathbb{R}} x(\exp(-\bar{\gamma}(t,T,z)) - 1)r(t,z)\nu(dz)
\end{aligned} \tag{5.99}$$

Since this is a linear expression in x, it is natural to assume for optimal $(\hat{p},\hat{q},\hat{r})$ that $\frac{dH}{dx} = 0$, which means that

$$\begin{aligned}
&-\bar{\alpha}(t,T) + \frac{1}{2}\bar{\sigma}^2(t,T) + \int_{\mathbb{R}} (\exp(-\bar{\gamma}(t,T,z)) \\
&-1 + \bar{\gamma}(t,T,z))\nu(dz))\hat{p} - \bar{\sigma}(t,T)\hat{q} \\
&+ \int_{\mathbb{R}} (\exp(-\bar{\gamma}(t,T,z)) - 1)\hat{r}(t,z)\nu(dz) = 0
\end{aligned} \tag{5.100}$$

Our adjoint equation will then be:

$$\begin{aligned}
dp(t) &= \Big[(\bar{\alpha}(t,T) - \frac{1}{2}\bar{\sigma}^2(t,T) - \int_{\mathbb{R}} (\exp(-\bar{\gamma}(t,T,z)) - 1 \\
&\quad + \bar{\gamma}(t,T,z))\nu(dz))p(t) + \bar{\sigma}(t,T)q(t) \\
&\quad - \int_{\mathbb{R}} (\exp(-\bar{\gamma}(t,T,z)) - 1)r(t,z)\nu(dz)\Big] dt + q(t)dB(t) \\
&\quad + \int_{\mathbb{R}} r(t^-,z)\tilde{N}(dt,dz) \\
p(T) &= \exp(-X_x^\pi(T))
\end{aligned} \tag{5.101}$$

and we see for optimal $(\hat{p}, \hat{q}, \hat{r})$ that our drift will be equal to zero. To solve this we try with the process $p(t) = \phi_t \exp(-X_x^\pi(t))$, and we recall that $dX_x^\pi(t) = \pi(t^-) d\tilde{P}(t^-, T)$. Here we need that ϕ_t is $\mathcal{F}_t$-predictable, and of the form $d\phi_t = \phi_t' dt$, we also assume that $\phi_t \neq 0$, and we need $\phi_T = 1$. When we have this, we get that $dp(t)$ is given by:

$$\begin{aligned}
dp(t) \quad = \quad & -p(t)\pi(t)\tilde{P}(t,T)(-\bar{\alpha}(t,T) + \frac{1}{2}\bar{\sigma}^2(t,T) + \int_{\mathbb{R}} [\exp(-\bar{\gamma}(t,T,z)) \\
& -1 + \bar{\gamma}(t,T,z)]\nu(dz))dt + p(t)\pi(t)\tilde{P}(t,T)\bar{\sigma}(t,T)dB(t) \\
& +\frac{1}{2}p(t)(\pi(t)\tilde{P}(t,T)\bar{\sigma}(t,T))^2 dt + \int_{\mathbb{R}} p(t)\Big(e^{-\pi(t)\tilde{P}(t,T)(\exp(-\bar{\gamma}(t,T,z))-1)} - 1 \\
& +\pi(t)\tilde{P}(t,T)(\exp(-\bar{\gamma}(t,T,z)) - 1)\Big)\nu(dz)dt \\
& \int_{\mathbb{R}} p(t^-)\Big(e^{-\pi(t^-)\tilde{P}(t^-,T)(\exp(-\bar{\gamma}(t,T,z))-1)} - 1\Big)\tilde{N}(dt,dz) \\
& +e^{-X_x^\pi(t)}\phi_t' dt \qquad (5.102)
\end{aligned}$$

then we can compare (5.103) and (5.104) term for term, and we will get these equations:

$$\begin{aligned}
& (\bar{\alpha}(t,T) - \frac{1}{2}\bar{\sigma}^2(t,T) - \int_{\mathbb{R}} (\exp(-\bar{\gamma}(t,T,z)) - 1 + \bar{\gamma}(t,T,z))\nu(dz))p(t) \\
& +\bar{\sigma}(t,T)q(t) - \int_{\mathbb{R}} (\exp(-\bar{\gamma}(t,T,z)) - 1) r(t,z)\nu(dz) \\
= \quad & p(t)\pi(t)\tilde{P}(t,T)(\bar{\alpha}(t,T) - \frac{1}{2}\bar{\sigma}^2(t,T) - \int_{\mathbb{R}} [\exp(-\bar{\gamma}(t,T,z)) - 1 \\
& +\bar{\gamma}(t,T,z)]\nu(dz)) + \frac{1}{2}p(t)(\pi(t)\tilde{P}(t,T)\bar{\sigma}(t,T))^2 + \int_{\mathbb{R}} p(t)\Big(e^{-\pi(t)\tilde{P}(t,T)(\exp(-\bar{\gamma}(t,T,z))-1)} \\
& -1 + \pi(t)\tilde{P}(t,T)(\exp(-\bar{\gamma}(t,T,z)) - 1)\Big)\nu(dz) + e^{-\pi(t)\tilde{P}(t,T)}\phi_t' \qquad (5.103)
\end{aligned}$$

$$q(t) = p(t)\pi(t)\tilde{P}(t,T)\bar{\sigma}(t,T) \qquad (5.104)$$

$$r(t^-, z) = p(t^-)\Big(e^{-\pi(t^-)\tilde{P}(t^-,T)(\exp(-\bar{\gamma}(t,T,z))-1)} - 1\Big) \qquad (5.105)$$

then we can insert (5.104) and (5.105) into (5.103), and if these are optimal, we get from equation (5.100) that (5.103) is equal to 0, and we get this equation:

$$\begin{aligned}
&\Big[\bar{\alpha}(t,T) - \frac{1}{2}\bar{\sigma}^2(t,T) - \int_{\mathbb{R}} (\exp(-\bar{\gamma}(t,T,z)) - 1 + \bar{\gamma}(t,T,z))\nu(dz) \\
&+\bar{\sigma}(t,T)\pi(t)\tilde{P}(t,T)\bar{\sigma}(t,T) \\
&- \int_{\mathbb{R}} \Big(\exp(-\bar{\gamma}(t,T,z)) - 1\Big)\Big(e^{-\pi(t)\tilde{P}(t,T)(\exp(-\bar{\gamma}(t,T,z))-1)} - 1\Big)\nu(dz)\Big]p(t) \\
= \;& 0
\end{aligned} \tag{5.106}$$

and since $p(t) = \phi_t \exp(-X_x^\pi(t)) \neq 0$, we can rearrange this equation and divide by $p(t)$to attain this equation instead:

$$\begin{aligned}
&\bar{\alpha}(t,T) - \frac{1}{2}\bar{\sigma}^2(t,T) - \int_{\mathbb{R}} \bar{\gamma}(t,T,z)\nu(dz) \\
= \;& -\bar{\sigma}^2(t,T)\pi(t)\tilde{P}(t,T) + \int_{\mathbb{R}} e^{-\pi(t)\tilde{P}(t,T)(\exp(-\bar{\gamma}(t,T,z))-1)} \left(e^{-\bar{\gamma}(t,T,z)} - 1\right)\nu(dz)
\end{aligned} \tag{5.107}$$

Then we notice that this is the same equation as we got when we solved the MEMM, but with variable $\pi(t)\tilde{P}(t,T)$. Since we had a unique solution to the MEMM problem, we get that the solution to (5.107) and the solution from the MEMM problem is equal. We recall from the MEMM that we had the function $v(t)$ as a variable, and this was equal to $-\frac{u(t)}{\bar{\sigma}(t,T)}$, so we get this expression for $\pi(t)\tilde{P}(t,T)$:

$$\pi(t)\tilde{P}(t,T) = -\frac{u(t)}{\bar{\sigma}(t,T)} \tag{5.108}$$

this is the same expression as we got when we used the dual formulation. Then the last thing we need is the process ϕ_t, and this solves the differential equation:

$$\begin{aligned}
\phi_t' \;=\; & -\phi_t\Big\{\pi(t)\tilde{P}(t,T)(\bar{\alpha}(t,T) - \frac{1}{2}\bar{\sigma}^2(t,T) - \int_{\mathbb{R}}[\exp(-\bar{\gamma}(t,T,z)) - 1 \\
&+\bar{\gamma}(t,T,z)]\nu(dz)) + \frac{1}{2}p(t)(\pi(t)\tilde{P}(t,T)\bar{\sigma}(t,T))^2 + \int_{\mathbb{R}}\Big(e^{-\pi(t)\tilde{P}(t,T)(\exp(-\bar{\gamma}(t,T,z))-1)} \\
&-1 + \pi(t)\tilde{P}(t,T)(\exp(-\bar{\gamma}(t,T,z)) - 1)\Big)\nu(dz)\Big\}
\end{aligned} \tag{5.109}$$

with boundary condition $\phi_T = 1$, and the solution to this is:

$$\begin{aligned}\phi_t &= \exp\Bigg(\int_t^T \Big\{\pi(s)\tilde{P}(s,T)(\bar{\alpha}(s,T) - \frac{1}{2}\bar{\sigma}^2(s,T) - \int_{\mathbb{R}}[\exp(-\bar{\gamma}(s,T,z)) - 1 \\ &\quad +\bar{\gamma}(s,T,z)]\nu(dz)) + \frac{1}{2}p(s)(\pi(s)\tilde{P}(s,T)\bar{\sigma}(s,T))^2 + \int_{\mathbb{R}}\Big(e^{-\pi(s)\tilde{P}(s,T)(\exp(-\bar{\gamma}(s,T,z))-1)} \\ &\quad -1 + \pi(s)\tilde{P}(s,T)(\exp(-\bar{\gamma}(s,T,z)) - 1)\Big)\nu(dz)\Big\}ds\Bigg) \qquad (5.110)\end{aligned}$$

So our conclusion here is that we get our same investment strategy here as we did when we used the duality method, and this means we will also get the same final wealth. Our conclusion from this is that the Inada conditions on the utility function $U(x)$:

$$\begin{aligned}U'(0) &= \lim_{x\to 0^+} U'(x) = \infty \\ U'(\infty) &= \lim_{x\to\infty} U'(x) = 0\end{aligned}$$

is not necessary to get the connection between the optimal measure and the optimal investment strategy. We also note that what we find here is what we invest in the risky asset, what we have to invest in the bank account will be found by using the self-financing property.

We also note that this duality method is used when we have a discounted wealth process, and it might be more natural to maximize the wealth instead. Since what we do is to look at the measure such that the discounted bond price $\tilde{P}(t,T)$ is a martingale, and optimize this, it should be possible to look at the measure that makes $P(t,T)$ a martingale, and optimize this to get a result for the wealth process instead.

6 Conclusion and Further Research

6.1 Conclusion

The topic of this thesis has in general been to extend the theory of interest rates from a market driven by a Brownian motion to a market driven by jump diffusions. The new theory in this thesis has in general been shown in Chapter 4 and 5, and we can separate our results into four parts.

(1) In Chapter 4 we introduce our new model and we find certain characteristics and requirements on this model, but the only new result is the equation we find for no arbitrage by the Girsanov theorem.
(2) In Chapter 5 we found most our results. First we concentrated on how we could find different martingale measures for our HJM model. These measures were found by either the maximum principle or the Esscher transform, and the main result here is showing the equality between the Minimal Entropy Martingale Measure and the Simple Return Esscher Transformed Martingale Measure.
(3) In Section 5.4 we focused on the European call option, and we showed how we could compute a price for this option using Fourier transforms. Our result is for a general measure change, if we want a specific option price we need to insert a specific measure change. While we only show how this is used for the European call option, we can generalize this result for other options by using another function in our Fourier transform.
(4) At last we look at investment strategies in Section 5.5 and our result here is an investment strategy that maximizes the expected utility when we have an exponential utility function. Our result here is first found with a duality method, and then we compare this with what we get in a straight forward computation and we see we get the same result. This implies that the Inada conditions that is needed for the duality method is not always necessary.

6.2 Further Research

Now we have summarized what we have found out in this thesis, but there is still a lot to do before we can use the results in this thesis for computations of real option prices and to find real investment strategies. A part of what is left can be summarized in these four points:

(1) In this thesis we find different martingale measures, but while our equa-

tion for no arbitrage holds for random processes, our specific measures only hold when our integrands are deterministic. So further research may include seeing what happens for general integrands.
(2) A large part of this thesis is to find different martingale measures, but our measures are only described by different equations that need to hold, we have no analytic solutions, we can only show that solutions exists. To find solutions we will need to know how the Lévy measure ν is defined, and even then it is not given we can find an analytic solution, and we might need numerical methods to find these solutions. We will also get the same problem for our investment strategies.
(3) In our computation for our European call option price we do not end up with a number, but we get an integral, and this integral need to be computed to find our final result. To compute this integral we will probably need numerical methods that are beyond the scope of this thesis, and we need to show that this integral converge since we have an integral over a line that is parallel with $\mathbb{R}$.
(4) At last we have the problem that we only have general integrands, with some restrictions, in this thesis, to use the results we will need specific functions and to find these we will need to fit these functions to historical data. We will probably need to have these specific functions before we can solve our other problems as well.

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